Review Problems for Midterm Exam II – MTH 299 Fall 2014

1. *Disprove the statement*: There is a real root of equation $\frac{1}{5}x^5 + \frac{2}{3}x^3 + 2x = 0$ on the interval $(1, 2)$.

2. *Prove:* there exists a real number x such that $\dfrac{x^2 + 3x - 3}{2x + 3} = 1$.

3. Let $f(x) = x^3 - 3x^2 + 2x - 4$. Prove that there exists a real number r such that $2 < r < 3$ and $f(r) = 0$.

4. Prove that there is no smallest positive rational number.

5. Prove that there is no largest prime.

   Hint: Use proof by contradiction and consider $n = p_1 ... p_k + 1$, where $p_i, i = 1, ..., k$ are all the possible prime numbers, as per your assumption.

6. Let x be an irrational real number. Prove that either x^2 or x^3 is irrational.

7. Prove that $\sqrt{5}$ is irrational. (You can use the fact that $5 \mid x^2$ if and only if $5 \mid x$.)

8. Prove that if $x, y \in \mathbb{Z}$, then $x^2 - 4y \neq 2$.

9. Use induction to prove that

$$1 + 3 + 6 + \cdots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

   for all $n \in \mathbb{N}$.

10. Prove that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n+1}$$

    for all $n \in \mathbb{N}$ with $n \geq 3$. (Note that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} > 1 + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{4}} = 2$.)

11. Use the Strong Principle of Mathematical Induction to prove that for each integer $n \geq 13$, there are nonnegative integers x and y such that $n = 3x + 4y$.

12. Let R be a relation defined on $\mathbb{N}^2$ by $(a, b)R(c, d)$ if $ad = bc$. Prove or disprove that a relation R is an equivalence relation and describe the elements in the equivalence class $[(1, 2)]$.

13. For (a, b) and $(c, d) \in \mathbb{R}^2$, define $(a, b) \sim (c, d)$ if $\lfloor a \rfloor = \lfloor c \rfloor$ and $\lfloor b \rfloor = \lfloor d \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Prove or disprove that a relation $\sim$ is an equivalence relation in $\mathbb{R}^2$.

14. A relation R is defined on the set of positive rational numbers by aRb if $\frac{a}{b} \in \{3^k : k \in \mathbb{Z}\}$. Prove that a relation R is an equivalence relation and describe the elements in the equivalence class $[2]$.

15. (a) Fill in the following addition and multiplication tables for $\mathbb{Z}_4$.

$+$	$[0]$	$[1]$	$[2]$	$[3]$
$[0]$				
$[1]$				
$[2]$				
$[3]$				

$\times$	$[0]$	$[1]$	$[2]$	$[3]$
$[0]$				
$[1]$				
$[2]$				
$[3]$				

(b) For each of the following modular arithmetic equations, use the tables above to either find *all* solutions or explain why it has no solution. The coefficients and variables should be taken in $\mathbb{Z}_4$.

i. $[3]\,x + [1] = [2]$
ii. $[2]\,x + [2] = [3]$
iii. $[2]\,x + [1] = [3]$

16. Let $[a]\,,[b] \in \mathbb{Z}_5$ and $[a] \neq [0]$. Prove that the equation $[a]\,x + [b] = 0$ always has *exactly one* solution.

17. Let $[a]\,,[b] \in \mathbb{Z}_5$, and assume $[a] \cap [b] \neq \emptyset$. Prove that $[a] = [b]$.

18. Fill in the blanks in part(a).

(a) Let A and B be sets. A relation $R \subset A \times B$ defines a function from A to B if

(1) $\forall a \in A$, $\exists b \in B$ such that ________________ and

(2) $\forall a \in A$, $\forall b_1, b_2 \in B$, if $(a, b_1) \in R$ and $(a, b_2) \in R$, then

________________.

(b) Let A be a set. Prove that there exists a unique relation R on A such that R is an equivalence relation on A and R is a function from A to A.

19. Let A and B be sets. Suppose that $f : A \to B$ is a function. Let $C, D \subseteq B$. Prove that

$$f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D).$$

20. Let $\mathbb{Z}_7 = \{[0], [1], \ldots, [6]\}$ be the set of congruence classes of integers modulo 7 together with the operations of addition and multiplication of congruence classes. Suppose that $f : \mathbb{Z}_7 \to \mathbb{Z}_7$ is the function defined by the rule

$$f([x]) = [2x + 1] \text{ for each } x \in \mathbb{Z}.$$

(a) Prove that f is a bijection.

(b) Prove that there exists integers a and b such that f^{-1} is given by the rule

$$f^{-1}([x]) = [ax + b] \text{ for each } x \in \mathbb{Z}.$$

21. For the functions below determine whether (i) they are well defined. If so determine whether (ii) they are injective, (iii) they are surjective.

(a) $f : \mathbb{Q} \to \mathbb{Z}$ defined by $f(\frac{a}{b}) = a + b$ for $a, b \in \mathbb{Z}$, $b \neq 0$.

(b) $g : \mathbb{Z}_4 \to \mathbb{Z}_8$ defined by $g([x]_4) = [x]_8$, where $[x]_p$ denotes the congruence class of an integer x modulo p.

(c) $h : \mathbb{Z}_8 \to \mathbb{Z}_4$ defined by $h([x]_8) = [x]_4$, where $[x]_p$ denotes the congruence class of an integer x modulo p.

22. Show that the sets A and B are numerically equivalent (have the same cardinality) by constructing an explicit bijection between A and B and proving the function you constructed is indeed a bijection.

(a) $A = \mathbb{N}$, B is the set of positive odd integers greater than 100.

(b) $A = \mathbb{N}$, $B = \mathbb{Z} \setminus \{-10, -9, -8, ..., 8, 9, 10\}$.

(c) $A = [0, 1]$, $B = [10, 15]$.