1. Let $n \in \mathbb{N}$ and $I = \{1, 2, \dots, n\}$. For $i \in I$, define $A_i = [(i-1)/n, i/n]$. Identify each of the following sets by writing it as an interval or a union of two intervals.

   (a) $\bigcup_{i \in I} A_i$

   (b) $\bigcap_{i \in I} \overline{A_i}$,

   with complements taken in the universal set $\mathbb{R}$.

2. Consider the set $S = \{\emptyset, \square\}$.

   (a) List the elements of $\mathcal{P}(S)$.

   (b) List the elements of $\mathcal{P}(\mathcal{P}(S))$.

   (c) Find a partition of $\mathcal{P}(\mathcal{P}(S))$ into 3 sets.

   (d) Is it possible to find a partition of S into 3 sets? Explain.

3. Prove the statements appearing in (a)-(b), and answer the prompt in (c)-(d). The symbol $\equiv$ denotes *congruence* modulo n, where $n \in \mathbb{Z}$ such that $n \geq 2$.

   (a) For all $a, b \in \mathbb{Z}$, if $a \equiv b$, then $b \equiv a$.

   (b) For all sets $a, b, c \in \mathbb{Z}$, if $a \equiv b$ and $b \equiv c$, then $a \equiv c$.

   (c) State the negation of each of the statements (a)-(b) above. Determine if the negation is true or false. Provide a counterexample for any false statement.

   (d) Let $a, b, c \in \mathbb{Z}$, and consider the conditional statement

   P: If $a \equiv b$ and $b \equiv c$, then $a \equiv c$.

   State the inverse, contrapositve and converse of statement P. Determine whether each of these is true or false.

4. Negate the following.

   (a) $\forall n \in \mathbb{Z},\ \exists m \in \mathbb{Z},\ m \cdot n = 1$.

   (b) $\exists x \in \mathbb{Q},\ \forall y \in Q,\ x \cdot y = y$.

   Rewrite the statements in (a) and (b) without the use of the symbols $\forall, \exists$, and state whether each is a true or a false statement. If it is a true statement, prove it. If it is a false statement, provide a counterexample.

5. Construct a truth table to show that the contrapositive of $A \Rightarrow B$ is equivalent to $A \Rightarrow B$.

6. Prove the following statement.

$$\forall a \in \mathbb{R}\, \exists x \in \mathbb{R},\ 3x - 1 = a.$$

7. Let E denote the set of even integers, $x \in \mathbb{Z}$, and $A(x)$ be the following open sentence.

$$A(x) : \text{“}x \in E \Rightarrow \exists k \in \mathbb{Z} \text{ such that } x = 2k\text{”}$$

   (a) Write the inverse of $A(x)$.

   (b) Write the converse of $A(x)$.

   (c) Write the contrapositive of $A(x)$.

   (d) Is $A(x)$ true for all $x \in \mathbb{Z}$? What about its converse? In this case, how would you restate it using *necessary/ sufficient/ necessary and sufficient*?

8. Let $A = \{x \in \mathbb{Z} | x = 6k, k \in \mathbb{Z}\}$,
   $B = \{x \in \mathbb{Z} | x = 2k, k \in \mathbb{Z}\}$,
   $C = \{x \in \mathbb{Z} | x = 3k, k \in \mathbb{Z}\}$. Prove the following statement.

$$x \in A \iff (\exists y \in B\, \exists z \in C,\ x = yz)$$

9. Construct a truth table to show that $A \Rightarrow B$ is equivalent to the statement: not(A) or B.

10. Let $a, b, c \in \mathbb{R}$, and consider the following open sentence:

$P(a, b, c)$: A necessary condition for the equation $ax^2 + bx + c = 0$ to have a solution x is: $a \neq 0$ and $b^2 - 4ac \geq 0$.

   (a) Rephrase $P(a, b, c)$ as an if-then implication; explicitly write all relevant quantifiers.

   (b) Write the contrapositive.

   (c) Write the converse.

   (d) Write the inverse.

   (e) Write the negation of $P(a, b, c)$ (simplified by moving the *not* as far into the statement as possible).

   (f) Which of the above statements (a)-(d) are equivalent to each other (for all $a, b, c \in \mathbb{R}$)?

   (g) The statement '$\forall a, b, c \in \mathbb{R},\ P(a, b, c)$' is false. Disprove it (prove the negation).

11. Let $x, y \in \mathbb{R}$. Prove that $(x + y)^2 = x^2 + y^2$ if and only if $xy = 0$.

12. Let $n \in \mathbb{Z}$. Prove that n is odd if and only if $n + 7$ is even.

13. Let $n \in \mathbb{Z}$. Prove that n is odd if and only if n^2 is odd.

14. Let $a \in (0, \infty)$. Prove that a rectangle with perimeter $4a$ is a square if and only if its area is a^2.

15. Define the *Euclidean norm* of $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ by $||\mathbf{x}|| = \sqrt{x_1^2 + ... + x_n^2}$. Prove that $||\mathbf{x}|| = 0$ if and only if $(x_1, x_2, ..., x_n) = (0, 0, ..., 0)$.

16. Let $x \in \mathbb{Z}$.
    (a) Prove that $x^2 + x$ is even.
    (b) Assume $x \neq 0$. Prove that $(x^2 + x)/2$ is divisible by x if and only if x is odd.
    (c) Assume $x + 1 \neq 0$. Prove that $(x^2 + x)/2$ is divisible by $x + 1$ if and only if x is even.

17. Show that if $x^2 - 3x + 2 < 0$, then $1 < x < 2$.

18. Let $a, b, c, d \in \mathbb{Z}$ with a and b nonzero. Prove that if $ab \nmid cd$, then $a \nmid c$ or $b \nmid d$.

19. Prove that for any two sets A and B contained in the same universal set, $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$.

20. Prove that for any two sets A and B, $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

21. Prove that for any sets A, B and C, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

22. Prove that if $n|a$ then $n|a + b \Leftrightarrow n|b$

23. Let $A = \{a \in \mathbb{R} : |a - 1| \leq 1\}$ and let B be the interval $[0, 3]$. Give a geometric description of $A \times B$ as a subset of $\mathbb{R} \times \mathbb{R}$. Can you conclude $\overline{A \times B} = \overline{A} \times \overline{B}$?