Math 299 **Supplement: Sets and Functions** **Sep 4, 2013**

All mathematical objects are formally defined in terms of sets and lists. Also, real-world objects usually have a natural mathematical model in these terms.

DEFINITION: A *set* is any collection of objects (elements), either written out or specified by some condition. Order of elements is irrelevant, and no repeated elements are allowed. We enclose a set with curly brackets $\{\,\}$.

EXAMPLES:

- $A = \{1, 3, 5\} = \{3, 5, 1\}$.
- $\mathbb{N} = \{\text{all natural numbers}\} = \{0, 1, 2, 3, \ldots\}$.
- $\mathbb{R} = \{\text{all real numbers}\}$.
- $B = \{x \in \mathbb{R} \text{ such that } x^2 = 2\} = \{\sqrt{2}, -\sqrt{2}\}$.
- $C = \{x \in \mathbb{N} \text{ such that } x^2 = 2\} = \{\}$, the empty set, since there is no natural-number solution to $x^2 = 2$.
- $D = \{x \in \mathbb{R} \text{ such that } x^2 > 1\} = \{x \in \mathbb{R} \text{ such that } x > 1 \text{ or } x < -1\}$.

DEFINITION: A *list* is an ordered sequence of any objects (entries), with repeat entries allowed. We enclose a list with round parentheses ( ).

EXAMPLES:

- $(1, 3, 5) \neq (1, 1, 3, 5) \neq (1, 5, 1, 3)$.
- An infinite sequence is a list, such as the even numbers $a_n = 2n$:

$$(a_n)_{n=1}^{\infty} = (a_1, a_2, a_3, \ldots) = (2, 4, 6, \ldots)$$

DEFNITION: The *Cartesian product* $A \times B$ is the set of all pairs (a, b) with a drawn from A and b drawn from B:

$$A \times B = \{(a, b) \text{ for all } a \in A,\ b \in B\}.$$

We use the times symbol $\times$ because of the counting formula: $|A \times B| = |A| \cdot |B|$.

Sample definitions

Coordinate geometry. A point in the coordinate plane is given by a list of two real-number coordinates, $P = (x, y)$, such as the origin $(0, 0)$. The plane is just the set of all such points:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \text{ for all } x, y \in \mathbb{R}\}.$$

The line L through the points $(1, 0)$ and $(0, 1)$ is the set of all points (infinitely many) satisfying a certain linear equation:

$$L = \{(x, y) \in \mathbb{R}^2 \;\text{ s.t. }\; x + y = 1\}.$$

The points $(1, 0)$, $(\frac{1}{3}, \frac{2}{3})$, $(-\sqrt{2}, \sqrt{2}{-}1)$ are some elements of L.

Playing cards. Cards in a standard deck are distinguished by two pieces of data: a face value $\mathrm{A}, 2, 3, \ldots, 10, \mathrm{J}, \mathrm{Q}$, or K; and a suit $\spadesuit, \clubsuit, \heartsuit$, or $\diamondsuit$. It is natural to model a card as a pair list: e.g. the queen of spades corresponds to $c = (\mathrm{Q}, \spadesuit)$. Then the deck is the cartesian product $D = F \times S$, where:

$$F = \{\mathrm{A}, 2, 3, \ldots, 10, \mathrm{J}, \mathrm{Q}, K\} \qquad \text{and} \qquad S = \{\spadesuit, \clubsuit, \heartsuit, \diamondsuit\}.$$

Note that D is naturally a set rather than a list, because the deck remains the same regardless of how it is shuffled.

A hand H is an unordered collection of cards from the deck: that is, a subset $H \subset D$. To study poker odds, we put all possible 5-card hands into a set:

$$P = \{H \subset D \;\text{ s.t. }\; |H| = 5\}.$$

Notice that each element $H \in P$ is itself a set, and each element $c \in H$ is itself a list (a pair). All kinds of real-world objects can be modeled by some such layer-cake of sets and lists.

Ordered pair definition of a function. A function $f : A \to B$ is any "rule" taking each input $a \in A$ to an output $b = f(a) \in B$. But if two rules give identical outputs for each input, then they define the same function. To formalize this, we give a function by its "table of values":

$$f \;=\; \{(a, b) \in A \times B \;\text{ s.t. }\; b = f(a)\} \;=\; \{(a, f(a)) \text{ for all } a \in A\}.$$

For example, the function $f(n) = n^2 + n$ on natural numbers $n \in \mathbb{N}$ is defined by the table:

n	0	1	2	3	$\cdots$
$f(n)$	0	2	5	12	$\cdots$

which can be thought of as the set of all pairs of inputs and outputs:

$$\begin{aligned} f &= \{(0,0),(1,2),(2,5),(3,12),\ldots\} \\ &= \{(n, n^2{+}n) \text{ for all } n \in \mathbb{N}\}. \end{aligned}$$

Note that the function $g(n) = n(n+1)$ is defined by a different formula, but gives the same values, the same set of pairs, and hence the same function: $f = g$.

What is formal mathematics? The above set-and-list definitions seem to forget the "real meaning" of the objects they model. For example, a card is a marked piece of paper, not a pair like (Q, ♠). Yet this data identifies each card, so the formal definition is just what we need to model and solve any problem about card-hands. Similarly, the plane coordinates (x, y) don't describe *what* a geometric point is, but they determine *where* it is, and this is all that is relevant to any geometric problem.

For a function on the real numbers, the set-of-pairs definition gives the same formal object as a set of points in the plane: for example, the function $\ell(x) = -x + 1$ for $x \in \mathbb{R}$ gives:

$$\ell = \{(x, y) \in \mathbb{R}^2 \ \text{ s.t. } \ y = -x + 1\},$$

which is just the line L in a previous example. Why this coincidence? Because the line is the graph of the function! The function and its graph are defined by the *same data*, so formally, the function *is* its graph.

Although formal definitions in terms of sets and lists do not describe the "real meaning" of the objects, they contain all the identifying data. For *formal* reasoning, we work only with the set-theory data, and leave the intuitive meaning for *informal* discussion.