Math 299 **Homework assignment, due 11/15**

Consider the modular number system $\mathbb{Z}_9$

1. Write the complete 9×9 addition and mulitiplication tables. For example, we have $\bar{6} + \bar{7} = \bar{13} = \bar{4}$, so in the addtion table, the entry in the $\bar{6}$ row and $\bar{7}$ column should be $\bar{4}$. Hint: For simplicity, don't write the lines over the numbers in the table: just keep in mind that all the entries are classes in $\mathbb{Z}_9$, so that everything is modulo 9.

2. Looking at the multiplication table, determine which elements $\bar{a} \in \mathbb{Z}_9$ have inverses $\bar{a}^{\,-1}$.

3. Determine which elements have square roots. That is, for which $\bar{a} \in \mathbb{Z}_9$ is there some $\bar{b} \in \mathbb{Z}_9$ with $\bar{b}^2 = \bar{a}$?

4. Use the quadratic formula to solve the equation $x^2 + \bar{3}x + \bar{5} = \bar{0}$ for $x \in \mathbb{Z}_9$. (*For this part of the problem, see the discussion below.*)

Modular algebra. Since $\mathbb{Z}_p$ (for p a prime) obeys all the usual axioms of addition and multiplication, almost everything we know about algebra carries over to $\mathbb{Z}_p$, provided we remember that $\bar{p} = \bar{0}$.

For example, the quadratic formula gives the solutions to the equation $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,.$$

Now, if we want to solve an equation like:

$$x^2 + \bar{2}x + \bar{3} = \bar{0} \quad \text{for} \quad x \in \mathbb{Z}_{11}\,,$$

we apply the quadratic formula to the number system $\mathbb{Z}_{11}$. We need the square root of $b^{\,2} - 4ac = \bar{-8} = \bar{3}$, which by definition is some $y \in \mathbb{Z}_{11}$ with $y^2 = \bar{3}$. By trial and error we find $\bar{5}^2 = \bar{25} = \bar{3}$, so we take $y = \bar{5}$. Also, dividing by $2a = \bar{2}$ means multiplying by $\bar{2}^{\,-1} = \bar{6}$. Thus we get:

$$x = (-b \pm y)(2a)^{-1} = (-\bar{2} \pm \bar{5})(\bar{6}) = \bar{18}, \bar{-42} = \bar{7}, \bar{2}.$$

Check: for $x = \bar{7}$, we have: $(\bar{7})^2 + \bar{2}(\bar{7}) + \bar{3} = \bar{66} = \bar{0}$, and similarly for $x = \bar{2}$.