Supplement 7 for Section 4.4

Replace the definition on page 203 with the definition presented here.

Definition 1. Let f be defined on an interval I.
1. Then f is concave up on I means for each pair $a < b \in I$, the graph of f between a and b lies below the line segment joining $(a, f(a))$ and $(b, f(b))$.
2. Then f is concave down on I means for each pair $a < b \in I$, the graph of f between a and b lies above the line segment joining $(a, f(a))$ and $(b, f(b))$.

It should be obvious that f is concave up on an interval I if and only if $-f$ is concave down on I. What isn't so obvious is that a function can be concave up on an interval without being differentiable everywhere on that interval. For example the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ (x+1)^3 - 1 & \text{if } x > 0 \end{cases}$$

is concave up on $(-\infty, \infty)$ which can easily be seen by sketching its graph. However, $\lim_{x \to 0^-} \frac{f(x)-0}{x-0} = \frac{d}{dx} x^2 |_{x=0} = 0$ while $\lim_{x \to 0^+} \frac{f(x)-0}{x-0} = \frac{d}{dx}\big((x+1)^3 - 1\big)|_{x=0} = 1$. Consequently $f'(0)$ doesn't exist. But if f is differentiable, then it's possible to determine whether or not the function is concave up or down from the derivative as the following theorem indicates.

Theorem 1. *Let f be continuous on an interval I and differentiable on the interior of I.*
1. If f' is increasing on the interior of I, then f is concave up on I.
2. If f' is decreasing on the interior of I, then f is concave down on I.

The next test for concavity is a consequence of Corollary 3 on page 119.

Corollary 1. *Let f be continuous on an interval I and twice differentiable on the interior of I.*
1. If $f''(x) > 0$ for each x in the interior of I, then f is concave up on I.
2. If $f''(x) < 0$ for each x in the interior of I, then f is concave down on I.