# Affine Invariant Measures in Levi-Civita Vector Spaces and the Erdös Obtuse Angle Theorem

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ABSTRACT. An interesting question posed by Paul Erdös around 1950 pertains to the maximal number of points in n-dimensional Euclidean Space so that no subset of three points can be picked that form an obtuse angle. An unexpected and surprising solution was presented around a decade later. Interestingly enough the solution relies in its core on properties of measures in n-dimensional space. Beyond its intuitive appeal, the question can be used as a tool to assess the complexity of general vector spaces with Euclidean-like structures and the amount of similarity to the conventional real case.

We answer the question for the specific situation of non-Archimedean Levi-Civita vector spaces and show that they behave in the same manner as in the real case. To this end, we develop a Lebesgue measure in these spaces that is invariant under affine transformations and satisfies commonly expected properties of Lebesgue measures, and in particular a substitution rule based on Jacobians of transformations. Using the tools from this measure theory, we will show that the Obtuse Angle Theorem also holds on the non-Archimedean Levi-Civita vector spaces.

## 1. Introduction

In order to formulate the obtuse angle problem more clearly and put it into context, we begin with some observations about the matter at hand. First, let us formulate it in appropriate mathematical terminology. Let $V$ be a vector space over a totally ordered field $F$. Let $(\,,\,)$ denote an inner product, i.e. a function from $V^2 \to F$ that has the common linearity properties under vector addition and scalar multiplication on both sides. We say three points $p_0$, $p_1$ and $p_2$ form an obtuse angle at $p_0$ if the vectors $p_1 - p_0$ and $p_2 - p_0$ have negative inner product, i.e. if $(p_1 - p_0, p_2 - p_0) < 0$. We say the three points form an obtuse angle if any one of the three permutations form an obtuse angle. Furthermore, we say a set of $n$ points forms an obtuse angle if there are three points in the set that do so. Apparently this algebraic notation generalizes the concept of obtuse angles in elementary geometry and the well-known Euclidean vector spaces of $\mathbb{R}^d$.

Let us now provide some perspective on the matter of point sets admitting obtuse angles for the vector spaces $\mathbb{R}^d$ and the common inner product. We begin by observing that apparently in $\mathbb{R}^d$ it is always possible to find sets of $2^d$ points that only admit non-obtuse angles, namely by merely picking the corner points of the unit cube $[0,1]^d$. More specifically, because of the rotational symmetry of the unit cube, without loss of generality we can assume $p_0 = (0, ..., 0)$; and since any of

the other corner points have only non-negative components, the inner products of any two of them will always be positive or zero, and so no obtuse angles exist. The natural question now is whether $2^d$ is indeed the maximal number of any collection of points so that no obtuse angles exist, which was posed as a challenge by Paul Erdös [**9**].

To study this in some more depth, note that in the one dimensional case $\mathbb{R}$, any three distinct points form an obtuse angle; merely choose $p_0$ to be the middle point and observe that the differences $p_1 - p_0$ and $p_2 - p_0$ have opposite signs. In $\mathbb{R}^2$ it is easily possible to find three points that do not form obtuse angles; any subset of corner points of the unit cube will do, and so will the points in any equilateral triangle, etc. etc.

Now consider the case of five points in the plane. If these points are arranged so as to form a convex pentagon, i.e. every point forms a corner of the convex hull of the five points, then there must be at least one angle of $108°$ or more since the sum of angles in a pentagon is $540°$. On the other hand, let one point be inside the convex hull of the other four. Split the rectangle forming the convex hull of said four points into the union of two triangles and note that the interior point is inside at least one of these triangles. Drawing the connection lines of the interior point to the three corner points of the triangle we note that the angles between these connection lines add up to $360°$, so at least one of them is $120°$ or more. The situation in $\mathbb{R}^3$ is already significantly more complicated, and while elementary proofs exist, we forego their discussion.

In the following we will develop a proof of the theorem, trying to parallel the original result of Danzer and Grünbaum[**8**]. Various concepts need to be ported to Levi-Civita vector spaces, beginning the concepts of linear algebra, and most importantly, an extended measure theory for Levi-Civita vector spaces going beyond the natural generalization of the one dimensional measure based on intervals.

To provide the necessary foundations, we begin the discussion with an introduction of terminology and a review of some properties of totally ordered fields. Let $K$ be a totally ordered non-Archimedean field extension of the real numbers $\mathbb{R}$ and $\leq$ its order, which induces the $K$-valued absolute value $|\ |$. We use the following notation common to the study of non-Archimedean structures.

DEFINITION 1.1 ($\sim, \approx, \ll, H$). For $x, y \in K$, we say $x \sim y$ if there are $n, m \in \mathbb{N}$ such that $n \cdot |x| > |y|$ and $m \cdot |y| > |x|$ $x \ll y$ if for all $n \in \mathbb{N}$, $n \cdot |x| < |y|$, and $x \not\ll y$ if $x \ll y$ does not hold $x \approx y$ if $x \sim y$ and $(x - y) \ll x$.

We also set $[x] = \{y \in K | y \sim x\}$ as well as $H = \{[x] | x \in K\}$ and $\lambda(x) = [x]$.

Apparently the relation "$\sim$" is an equivalence relation; the set of classes $H$ of all nonzero elements of $K$ is naturally endowed with an addition via $[x]+[y] = [x \cdot y]$ and an order via $[x] > [y]$ if $x \ll y$, both of which are readily checked to be well-defined. The class $[1]$ is a neutral element, and for $x \neq 0$, $[1/x]$ is an additive inverse of $[x]$; thus $H$ forms a totally ordered group, often referred to as the Hahn group or skeleton group. The projection $\lambda$ from $K$ to $H$ satisfies $\lambda(x \cdot y) = \lambda(x) + \lambda(y)$ and is a valuation.

We say $x$ is infinitely larger than $y$ if $x \gg y$, $x$ is infinitely small or large if $x \ll 1$ or $x \gg 1$, respectively, and we say $x$ is finite if $x \sim 1$. For $r \in H$, we say $x =_r y$ if $\lambda(x - y) > r$; apparently, "$=_r$" is an equivalence relation.

The fundamental theorem of Hahn [**12**] (for more easily readable and modern versions see [**13**] as well as [**6**], [**7**], [**10**], [**11**][**27**], and also the overview in [**24**])

provides a classification of any non-Archimedean extension $K$ of $\mathbb{R}$ in terms of its skeleton group $H$. In fact, invoking the axiom of choice it is shown that the elements of $K$ can be written as generalized formal power series over the group $H$ with real coefficients, and the set of appearing "exponents" forms a well-ordered subset of $H$.

Particular examples of the large variety of such fields are the quotients of polynomials as the smallest totally ordered non-Archimedean field, and the formal Laurent series as the smallest non-Archimedean field that is Cauchy-complete, both of which have the integers $\mathbb{Z}$ as Hahn group. The rationals $\mathbb{Q}$ form the Hahn group of the quotients of polynomials with rational exponents, as well as the Puiseux series, which form the smallest algebraically closed non-Archimedean field; see for example [5], [25], [26], [14], [29]. In general, the algebraic properties of fields of formal power series have been rather extensively studied (see for example [28]), and there are various general theorems pertaining to algebraic closure and other properties [20], [26] which mainly rest on divisibility of the Hahn group.

In this paper we develop a measure theory on vector spaces of such fields, and it will turn out to be important that the structure be Cauchy complete. This entails that convergence of sequences and series has some unusual properties [2]; in fact, the series $\sum_{n=0}^{\infty} a_n$ converges if and only if its associated sequence $(a_n)$ is null; and in this case, the series even converges absolutely. In particular, it follows that power series $\sum_{n=0}^{\infty} a_n x^n$ with real coefficients converge if and only if the geometric sequence $x^n$ converges. Apparently for this to happen it is not sufficient that $x$ be less than 1 in magnitude; in fact, the geometric sequence diverges for any finite or infinitely large $x$.

However, for many of the further arguments, in particular pertaining to the continuation of real and complex analytic functions, we would like to assure that the sequence converges as long as $x$ is infinitely small; using that $\lambda(x^n) = n\lambda(x)$, this is apparently the case if the Hahn group $H$ is Archimedean. We summarize this in the following definition.

DEFINITION 1.2. **(Levi-Civita Field)** We call the non-Archimedean field $K$ a Levi-Civita field and denote it by $\mathcal{R}$ if it is Cauchy complete, and its Hahn group is Archimedean and divisible.

For the sake of simplicity, we also call the adjoint field of "complex-like" numbers $\mathcal{R} + i\mathcal{R}$, where $i$ is the imaginary unit, a Levi-Civita field, and denote it by $\mathcal{C}$. On $\mathcal{C}$, we set $|a + ib| = |a| + |b|$ (without too much difficulty, one can see that also the more conventional norm based on the root of squares of real and imaginary parts can be introduced), and $\lambda(a + ib) = \lambda(|a + ib|)$.

The original definition of the field described by Levi-Civita [17], [18], which we shall briefly outline, is indeed more limited. However, as shown in [4], the original Levi-Civita field represents the smallest example to our wider class of fields, and has the distinction of being the only one that is computationally treatable [3], [31]. It is based on the concept of families of left-finite sets, and their properties are discussed in detail in [1], [2].

Levi-Civita himself succeeded to show that his structure forms a totally ordered field that is Cauchy complete, and that any power series with real or complex coefficients converges for infinitely small arguments. By doing so, he succeeded to extend infinitely often differentiable functions into infinitely small neighborhoods by virtue of their local Taylor expansion. He also succeeded to show that the resulting extended functions are infinitely often differentiable in the sense of the

order topology, and on the original real points, their derivatives agree with those of the underlying original function. The subject appeared again in the work by Ostrowski [**23**], Neder [**22**], and later in the work of Laugwitz [**15**]. Two more recent accounts of this work can be found in the book by Lightstone and Robinson [**19**], which ends with the proof of Cauchy completeness, as well as in Laugwitz' account on Levi-Civita's work [**16**], which also contains a summary of properties of Levi-Civita fields.

In [**2**] it is shown by explicit construction that $\mathcal{C}$ is algebraically closed, and that $\mathcal{R}$ is real-closed, which also follows from general valuation theory, and specifically for example the work of Rayner[**26**]. Compared to the general Hahn fields, the Levi-Civita fields are characterized by well-ordered exponent sets that are particularly "small", and indeed minimally small to allow simultaneously algebraic closure and the Cauchy completeness, as shown in [**4**].

## 2. Measure Theory on Levi-Civita vector spaces

Attempts to formulate meaningful measure theories on non-Archimedean fields have to necessarily follow modified approaches rather than the common method of Lebesgue. The total disconnectedness of these spaces under the order topology, the lack of existence of suprema and infima of bounded sets, and the different orders of magnitude that exist in the non-Archimedean structures, prevent the use of concepts of measure theory for example on Banach spaces. We begin our discussion with several general observations related to the introduction of measures on Levi-Civita vector spaces. First we establish that the situation is indeed fundamentally different from the real case.

PROPOSITION 2.1. *There is no non-trivial translation invariant measure on the Levi-Civita field $\mathcal{R}$ or the vector spaces $\mathcal{R}^d$.*

PROOF. We follow an indirect argument. Suppose there is a non-trivial measure $m$ in $\mathcal{R}$. Let $A \subset \mathcal{R}$ be any bounded set with non-vanishing measure. Let $b \in \mathcal{R}$ be a bound of $A$, i.e. $x \in A \Rightarrow |x| \leq b$. Now consider the family of translates of $A$ as follows:

$$A_n = n \cdot \frac{b}{\delta} + A$$

where $\delta$ is an arbitrary positive infinitely small number. By translation invariance we have $m(A_n) = m(A)$, and by the boundedness of $A$ by $b$, we also have $A_i \cap A_j = \emptyset$ for $i \neq j$. Now consider the set

$$B = [-\frac{b}{\delta}, +\frac{b}{\delta}].$$

Apparently we have $A_n \subset B$ for all $n \in \mathbb{N}$. Because of the Archimedicity of $\mathbb{R}$ and because $m(B)$ is finite due to the measure being non-trivial, there exists $k \in \mathbb{N}$ such that $k \cdot m(A) > m(B)$, so we have on the one hand

$$m \left( \bigcup_{i=1}^{n} A_i \right) > m(B)$$

but on the other hand we also have

$$\bigcup_{i=1}^{n} A_i \subset B$$

and thus a contradiction. The argument can be carried out analogously in the vector spaces $\mathcal{R}^d$. □

Apparently the above problems could be remedied by allowing the values of the measures of the sets $A$ and $B$ in the above argument to themselves lie in the non-Archimedean field $\mathcal{R}$, which allows the scaling by infinitely large and infinitely small scaling factors. In such an approach, in the one-dimensional case of $\mathcal{R}$, any interval $(a, b)$ is assigned the non-Archimedean measure $b - a$. In fact, we observe that it is also not possible to utilize any smaller field than the field $\mathcal{R}$. Because if the measure of the set $[0, 1]$ is $c$, then scale invariance implies that the measure of $[0, 1/c]$ is 1; and further, for any $r \in \mathcal{R}$, we have $m([0, r/c]) = r$, and so it is apparently necessary that the range of all measures cannot consist of a set smaller than $\mathcal{R}$.

However, problems with a conventional measure theory immediately arise due to the non-existence of infima in $\mathcal{R}$, which play such a crucial role in the conventional definition of outer measures. Furthermore, as we shall see, in the higher dimensional case the simple product of intervals does not carry very far.

Perhaps the most robust way to define a measure in the Levi-Civita field $\mathcal{R}$ is based on the following idea which is studied in [**32**]:

DEFINITION 2.2. *Let $A \subset \mathcal{R}$ be given. Then we say that $A$ is measurable if for every $\epsilon > 0$ in $\mathcal{R}$, there exist a sequence of mutually disjoint intervals $(I_n)$ and a sequence of mutually disjoint intervals $(J_n)$ such that $\cup_{n=1}^{\infty} I_n \subset A \subset \cup_{n=1}^{\infty} J_n$, $\sum_{n=1}^{\infty} l(I_n)$ and $\sum_{n=1}^{\infty} l(J_n)$ converge in $\mathcal{R}$, and $\sum_{n=1}^{\infty} l(J_n) - \sum_{n=1}^{\infty} l(I_n) \leq \epsilon$.*

Let us analyze the definition for a moment. At first glance, the requirement that an inner approximation and an outer approximation differ by less than $\epsilon$ looks very familiar and appears natural as in the real case. However, a significant difference arises in the restrictions imposed by the required and necessary strong convergence of the sums of lengths, and the need to eventually have the limits of these sums of lengths differ by infinitely small amounts.

This situation does not pose a significant problem in the one-dimensional case, other than that for example the standard Cantor set is not measurable[**32**], but it does so in higher dimensions. To this end, let us consider higher dimensional extensions of definition 2.2 based on the use of cross products of intervals, or boxes; such structures have been studied in substantial detail in [**33**]. The difficulty is clearly identified with the following problem:

PROPOSITION 2.3. *(**Non-Measurability of Standard Triangle**) Let $T \subset \mathcal{R}^2$ be the triangle with corner points $(0,0)$, $(1,0)$ and $(0,1)$. Then this triangle is not measurable under the natural extension of the one dimensional measure to $\mathcal{R}^2$.*

PROOF. Assume it were, and pick $\epsilon$ positive and infinitely small. This requires that there is a lower sum of boxes $\sum_{n=1}^{\infty} l(I_n^x) \cdot l(I_n^y)$ and an upper sum of boxes $\sum_{n=1}^{\infty} l(J_n^x) \cdot l(J_n^y)$ that have an area that differs by less than $d$. Furthermore, since each sum converges, there is an $N$ such that all terms $l(I_n^x) \cdot l(I_n^y)$ and $l(J_n^x) \cdot l(J_n^y)$ are less than $d$ for $n > N$. Now project the various intervals to the real line, i.e. consider

$$\bar{I}_n^{x,y} = I_n^{x,y} \cap \mathbb{R} \text{ and }, \bar{J}_n^{x,y} = J_n^{x,y} \cap \mathbb{R}.$$

This entails that actually $l(\bar{I}_n^x) \cdot l(\bar{I}_n^y) = 0$ and $l(J_n^x) \cdot l(\bar{J}_n^y) = 0$ for $n > N$ and that we have

$$\sum_{n=1}^{N} l(\bar{I}_n^x) \cdot l(\bar{I}_n^y) = m(T \cap \mathbb{R}) = \sum_{n=1}^{N} l(\bar{J}_n^x) \cdot l(\bar{J}_n^y)$$

where $m(T \cap \mathbb{R})$ denotes the conventional measure on $\mathbb{R}^2$. However, it is an elementary result of real measure theory that it is not possible to decompose the real standard triangle into finitely many rectangles. Rather, as is well known, the fact that the difference between outer and inner box enclosures can be made smaller than any positive real $\epsilon$ directly requires the use of more and more boxes the smaller $\epsilon$ becomes.                                                    □

So it appears advisable to modify the definitions of measures for the Levi-Civita vector spaces $\mathcal{R}^d$; and if it is our goal to reproduce a proof of the obtuse angle theorem which as we shall see requires the ability to measure sets that are significantly more than polytopes, such an approach is essential.

It is well known that abstract measure theory on general Banach spaces greatly benefits from the notion of the $\sigma$-algebra, a collection of non-empty subsets of the space of interest $X$ such that for any countable family of sets within the collection, their union and intersection is also in the collection (see for example [**30**] and countless, but not countably many, other good introductory treatments of measure theory). The elements of the $\sigma$-algebra then form the measurable sets; and one quite readily obtains the numerous well-known properties of the measure. Furthermore, if the space $X$ has arithmetic structure, one may naturally demand that the $\sigma$-algebra is invariant under translations and scaling, or more generally, under affine transformations.

As we have seen above, this approach breaks down in Levi-Civita vector spaces: if the family of measurable sets contains all elementary rectangles with real end points, it should also contain the countable unions of them that make the standard triangle measurable. But as we have seen, precisely this is not possible. As a remedy, we consider a smaller family of measurable sets, and then recover the ability to deal with countable unions of a specific form retroactively.

DEFINITION 2.4. (**S-Algrebra of Measurable Sets**) Let $\mathcal{B}$ be a collection of subsets of $\mathcal{R}^d$. We say that $\mathcal{B}$ is an S-algebra if with $A_1, ..., A_n \in \mathcal{B}$, also $A_1 \cup ... \cup A_n \in \mathcal{B}$, and $A_1 \cap ... \cap A_n \in \mathcal{B}$.

Utilizing the S-algebra, it will be possible to introduce a measure as an extension of the one dimensional case definition 2.2, where intervals and their multi-dimensional equivalents will be replaced by elements of the S-algebra.

In the following section we will introduce a specific S-algebra, of which we show that it is the smallest affine invariant S-algebra containing the unit cube. Subsequently we will use this S-algebra to define an affine invariant measure on $\mathcal{R}^d$ and show some of its properties. We will conclude with the proof of the obtuse angle theorem.

## 3. The S-Algebra of Simplexes

We begin our discussion with the definition and properties of simplexes in the non-Archimedean Levi-Civita vector spaces $\mathcal{R}^d$, which will serve as the building

blocks of the subsequent work. In all of the subsequent steps, it is of prime importance that all algebraic manipulations and arguments can be carried out in the respective Levi-Civita fields, which follows from the known theory of their behavior.

DEFINITION 3.1. **(The Simplex and its Measure)** A set $S \subset \mathcal{R}^d$ is called a simplex spanned by the vectors $(v_0, ..., v_d)$ if

$$S = \{x \in \mathcal{R}^d : x = v_0 + t_1(v_1 - v_0) + ... + t_d(v_d - v_0), \sum_{i=1}^{d} t_i \leq 1, t_i \geq 0\}.$$

The measure $m(S) \in \mathcal{R}$ of the simplex is defined as

$$m(S) = \frac{1}{d!} \cdot |det(v_1 - v_0, ..., v_d - v_0)|$$

where the determinant in $\mathcal{R}^d$ is defined through the algebraic expression corresponding to those in the Real case. We call the simplex $P$ degenerate if its measure vanishes, otherwise we call it non-degenerate.

Before we proceed further, let us make several comments.

REMARK 3.2. We observe that the definition of the measure of the simplex in $\mathcal{R}^d$ parallels the common Riemann- and Lebesgue measures of a simplex in the conventional Euclidean space $\mathbb{R}^d$. Because of the basic properties of arithmetic in $\mathcal{R}$, in a fully analogous way as in the real case, it is easy to show that the common well-known properties about determinants hold, in particular relating to interchange of columns, performing linear combinations of columns, determinants of products of matrices, etc.

We also note that the definition of the simplex in dimension $d$ requires $(d + 1)$ vectors to define its corners. However, any closed hull of $(k + 1)$ vectors with $k < d$ also forms a simplex; one can just add $(d - k)$ copies of the first of the vector to arrive at the required total of $(d + 1)$ vectors, which does not affect the set $P$ described by the resulting closed hull. However, simple rules about determinants that carry over fully to the non-Archimedean spaces show that all such simplexes are degenerate and have zero measure.

Finally, to simplify further notation, in the following we do not distinguish between the simplex containing its closure as defined above, or the simplex without its closure, or the simplex with only part of its closure. This is in full agreement to the case of $\mathbb{R}$, where for the purposes of measure theory, we do not distinguish between the closed interval, the open interval, or the interval containing only one of its bounds.

LEMMA 3.3. *(Permutation of Vectors)* *Let $P$ be a simplex spanned by* $(v_0, ..., v_d)$. *Let $\pi$ be a permutation of $(0, 1, ..., d)$. Let $P'$ be the simplex given by the vertices $(v_{\pi(0)}, v_{\pi(1)}, ..., v_{\pi(d)})$. Then $P = P'$, and $m(P') = m(P)$.*

PROOF. We first consider the case of permutations of $(0, 1, ..., d)$ that leave "0" fixed, i.e. that satisfy $\pi(0) = 0$. Let $x \in P$, i.e. there exist positive $t_i$, $i = 1, ..., d$, with $\sum_{i=1}^{d} t_i \leq 1$ such that $x = v_0 + t_1(v_1 - v_0) + ... + t_d(v_d - v_0)$. Then by commutativity of addition in $\mathcal{R}$, we also have $x = v_0 + t_{\pi(1)}(v_{\pi(1)} - v_0) + ... + t_{\pi(d)}(v_{\pi(d)} - v_0)$, and since all $t_{\pi(i)}$ are non-negative and clearly $\sum_i t_{\pi(i)} \leq 1$, we have that $x \in P'$. Replacing $\pi$ by its inverse, we see $x \in P'$ implies $x \in P$, and so $P = P'$. We further see that in this case, $m(P') = m(P)$, since interchanging columns in matrices does not affect the absolute value of the determinant.

Now we consider the interchange of $v_0$ and $v_1$ and note that any arbitrary permutation of $(0, 1, ..., d)$ can be written as a composition of a permutation leaving 0 unchanged, and one that exchanges only 0 and 1; so that this concludes our proof for a general permutation $\pi$. We observe that for a point $x$ inside the simplex $P$,

$$
\begin{aligned}
x &= v_0 + t_1(v_1 - v_0) + t_d(v_d - v_0) + ... + t_d(v_d - v_0) \\
&= v_1 + (1 - t_1) \cdot (v_0 - v_1) + t_2(v_2 - v_0) + ... + t_d(v_d - v_0) \\
&= v_1 + (1 - t_1) \cdot (v_0 - v_1) + t_2(v_2 - v_1) - t_2 \cdot (v_0 - v_1) + ... + t_d(v_d - v_1) - t_d(v_0 - v_1) \\
&= v_1 + (1 - t_1 - t_2 - ... - t_d) \cdot (v_0 - v_1) + t_2 \cdot (v_2 - v_1) + ... + t_d(v_d - v_1)
\end{aligned}
$$

Now we study the new coefficients $s_1 = (1 - t_1 - t_2 - ... - t_d)$, $s_2 = t_2$, ... , $s_d = t_d$. Because of the respective conditions on the $t_i$, each of them is non-negative, and clearly their sum is bounded above by 1. And we have

$$
\begin{aligned}
m(P) &= \frac{1}{d!} \cdot |det(v_1 - v_0, v_2 - v_0, ..., v_d - v_0)| \\
&= \frac{1}{d!} \cdot |det(v_0 - v_1, v_2 - v_0, ..., v_d - v_0)| \\
&= \frac{1}{d!} \cdot |det(v_0 - v_1, v_2 - v_0 - (v_1 - v_0), ..., v_d - v_0 - (v_1 - v_0))| \\
&= \frac{1}{d!} \cdot |det(v_0 - v_1, v_2 - v_1, ..., v_d - v_1)| = m(P'),
\end{aligned}
$$

completing the proof.  □

REMARK 3.4. (Alternate Simplex Representation) Now that it is established that the vector $v_0$, which appears in a prominent role in the definition of the measure of the simplex, can be replaced by any other of the $(d + 1)$ vectors, we often also write the simplex in the following apparently equivalent version, which more directly visualizes the interchangeability of the vectors:

$$
S = \{x \in \mathcal{R}^d : x = t_0 v_0 + t_1 v_1 + ... + t_d v_d, \sum_{i=1}^{d} t_i = 1, t_i \geq 0\}.
$$

Before we proceed any further, it is useful to establish some basic properties of simplexes that prove useful for further discussion:

THEOREM 3.5. *(Properties of Simplexes)*

*(a) Every simplex is the affine image of the unit simplex $S = \{t_0 \cdot 0 + t_1 e_1 + ... + t_d e_d | \sum t_i = 1, t_i \geq 0\}$ spanned by the unit vectors $e_i$ of the space. If the simplex is non-degenerate, the affine transformation is a bijection.*

*(b) Any two non-degenerate simplexes are isomorphic images of each other under an affine transformation.*

*(c) Let the simplexes $S_{1,2}$ be related by an affine transformation via $S_2 = a + M(S_1)$, where $M$ is a linear transformation, then $m(S_2) = |\det(M)| \cdot m(S_1)$, where "det" denotes the determinant and "$| |$" denotes the absolute value.*

*(d) Every non-degenerate simplex is the intersection of $(d + 1)$ half spaces.*

PROOF. (a) Indeed the transformation from the unit simplex to the simplex of interest is given by the translational part $v_0$ and the matrix $M = (v_1, v_2, ..., v_d)$ containing the defining vectors of the simplex as columns. If the simplex is non-degenerate, then $M$ is invertible, and so is the affine transformation.

(b) Because of the non-degeneracy of the first simplex and (a), there is an affine transformation $A_1$ of the first simplex to the unit simplex. Composing this with the affine transformation $A_2$ from the unit simplex to the second simplex, we obtain the desired affine transformation. Because the determinants are nonzero, it is a bijection.

(c) This follows directly from the definition of the measure of the simplex.

(d) Apparently the unit simplex is the intersection of the $d$ positive half spaces and that half space spanned by the endpoints of the unit vectors that contains the origin. Observe that affine images of half spaces are again half spaces, and use (a) to obtain the half planes forming the general simplex. $\square$

We now proceed to more general objects that form the core of the further discussion.

DEFINITION 3.6. (**Convex Polytope**) Let $v_0$, $v_1$, ... , $v_p$ be vectors in $\mathcal{R}^d$. Then we call

$$P = \{x \in \mathcal{R}^d : x = t_0 \cdot v_0 + t_1 \cdot v_1 + ... + t_p \cdot v_p, \sum_{i=1}^{p} t_i = 1, t_i \geq 0\}$$

the convex span of the vectors $v_0$ to $v_p$ and denote it by $P(v_0, ..., v_p)$ or conv$\{v_0, ..., v_p\}$. Any set that can be written as such a convex span is called a convex polytope or simply polytope.

REMARK 3.7. Apparently the set $P$ is indeed convex, and simplexes are convex polytopes. Furthermore, similar to the case of the simplex, for notational simplicity we do not distinguish between the closed polytope containing its boundary, the open polytope, or any set in between these two. Furthermore, we say two polytopes are almost disjoint if their interiors are disjoint.

For the further discussion, we need some tools; in particular, we need to characterize a convex polytope by its vertices.

DEFINITION 3.8. Let $P$ be a convex polytope in $\mathcal{R}^d$ and $z \in P$. Then we call $z$ a vertex if $z = t \cdot x + (1 - t) \cdot y$ for some $0 \leq t \leq 1$ and $x, y \in P$ implies $z = x$ or $z = y$.

So $z$ is a vertex iff any line segment entirely in $P$ that contains $z$ has $z$ as an end point.

LEMMA 3.9. *Let $P(v_0, ..., v_p)$ be a convex polytope and $c$ a vertex of $P$. Then there is $i \in \{1, ..., p\}$ such that $c = v_i$.*

PROOF. Indirect. Assume that $c \in P$ is a vertex, but that $c \neq v_i$ for all $i = 0, ..., p$. Since $c \in P$, we can write

$$c = t_0 v_0 + t_1 v_1 + ... + t_p v_p$$

with suitable $t_i$. Since by assumption $c$ is not one of the $v_i$, at least two of these $t_i$ need to be distinct from 0 and 1. If all $v_i$ with nonzero $t_i$ are equal, then $c = v_i$ and we have a contradiction. Otherwise, at least two of the $v_i$ are distinct; let these be $v_0$ and $v_1$. We set

$$x = 0 \cdot v_0 + (t_0 + t_1) \cdot v_1 + t_2 \cdot v_2 + ... + t_p \cdot v_p$$
$$y = (t_0 + t_1) \cdot v_0 + 0 \cdot v_1 + t_2 \cdot v_2 + ... + t_p \cdot v_p$$

We observe that $x, y \in P$, that $x \neq y$ because by construction $v_0 \neq v_1$, and $z = t_1/(t_0 + t_1) \cdot x + t_0/(t_0 + t_1) \cdot y$. Thus, picking $t = t_1/(t_0 + t_1)$, we obtain a contradiction; so $c$ must be one of the $v_i$. $\qquad\square$

As we shall see now, the vertices of the convex polytope play a special role similar to the basis vectors of a vector space, while the non-vertex vectors spanning the polytope are superfluous.

THEOREM 3.10. *Let $P$ be a convex polytope and $c_i$ for $i = 0, .., N$ its vertices. Then*

$$P = \{x \in \mathcal{R}^d : x = t_0 c_0 + t_1 c_1 + ... + t_N c_N, \sum_{i=0}^{N} t_i = 1, t_i \geq 0\}$$

PROOF. According to the last lemma, the vertices $c_i$ are necessarily included in the vectors spanning the polytope; but there may perhaps be others of the $v_i$, $i = N+1, ..., p$ that are necessary to span the polytope. Assuming that there are, we denote these by $v_{N+1}, ..., v_p$, and we have

$$P = \{x \in \mathcal{R}^d : x = t_0 c_0 + t_1 c_1 + ... + t_N c_N + t_{N+1} v_{N+1} + ... + t_p v_p, \sum_{i=0}^{p} t_i = 1, t_i \geq 0\}.$$

Now we will show that any $v_{N+i}$ that is not a vertex can be eliminated from consideration. Specifically, we show that for any $v_i$ that is not a vertex, there are $t_1$ through $t_N$ so that

$$v_{N+i} = t_0^{N+i} c_0 + ... + t_N^{N+i} c_N$$

with $t_j^i \geq 0$ that satisfy $\sum_{j=0}^{N} t_j^{N+i} = 1$. If this is the case, then for any $x \in P$, we have

$$x = t_0 c_0 + ... + t_N c_N + t_{N+1} v_{N+1} + ... + t_p v_p$$
$$= t_0 c_0 + ... + t_N c_N + t_{N+1} \left(t_0^{N+1} c_0 + ... + t_N^{N+1} c_N\right) + ... + t_p \left(t_0^p c_0 + ... + t_N^p c_N\right)$$
$$= \left(t_0 + t_{N+1} t_0^{N+1} + ... + t_p t_0^p\right) c_0 + ... + \left(t_N + t_{N+1} t_N^{N+1} + ... + t_p t_N^p\right) c_N.$$

So this entails that any $x \in P$ can indeed be expressed merely as a linear combination of the $c_i$ for $i = 0, ..., N$. Furthermore, each of the coefficients is a sum of non-negative terms and thus non-negative. Finally, the sum of the coefficients is indeed 1, which can most easily be seen in the second line, where the coefficients in each parentheses add up to 1.

Now we need to show that each of the $v_{N+i}$ can actually be expressed as an affine combination of the $c_i$ as stated above. We proceed inductively, and first consider the polytope spanned by $c_0, ..., c_N$, and compare with the span obtained by adding on more non-vertex point from $P$ denoted by $v_{N+1}$. Since $v_{N+1}$ is not a vertex, there are distinct points $x, y \in P$ such that $v_{N+1}$ lies on the connection line of $x$ and $y$. Without loss of generality, because of the convexity of $P$, we can move the farther of these two points closer to $v_{N+1}$ so that actually $v_{N+1} = 1/2(x + y)$. Now we write $x, y \in P$ in terms of the vectors $c_0, ..., c_N$ and $v_{N+1}$ as

$$x = t_0^x c_0 + ... + t_N^x c_N + t_{N+1}^x v_{N+1}$$
$$y = t_0^y c_0 + ... + t_N^y c_N + t_{N+1}^y v_{N+1}$$

where the coefficients are non-negative and add to 1. Note that we cannot have $t_{N+1}^x = 1$ or $t_{N+1}^x = 1$, because either case would force all other coefficients in

the respective line to vanish and would mean that $x = v_{N+1}$ or $y = v_{N+1}$, which contradicts the choice of $x$ and $y$. We now observe

$$v_{N+1} = \frac{1}{2}(x + y) = \frac{t^x_{N+1} + t^y_{N+1}}{2} \cdot v_{N+1} + \frac{t^x_0 + t^y_0}{2} \cdot c_0 + ... + \frac{t^x_N + t^y_N}{2} \cdot c_N$$

Now we solve for $v_{N+1}$, which is possible since $\left(t^x_{N+1} + t^y_{N+1}\right)/2 ¡ 1$, and obtain

$$v_{N+1} = \frac{\frac{t^x_0 + t^y_0}{2} \cdot c_0 + ... + \frac{t^x_N + t^y_N}{2} \cdot c_N}{1 - \frac{t^x_{N+1} + t^y_{N+1}}{2}}.$$

Note that because the coefficients in the expressions for $x$ and $y$ added up to 1, it follows that those of the $c_i$ on the right hand side of the above equation also add up to 1. Apparently they are also all non-negative; so we have shown that $v_{N+1}$ already appears in the span of $c_0, ..., c_N$. Now one proceeds inductively, observing that including $v_{N+2}$ does not change the span, until reaching $v_{N+1}$. So we see that the polytope is spanned by its vertices only. □

Thus indeed, the vertices play an eminently prominent role in the description of a polytope. Since they are unique up to re-ordering, they also allow a useful classification of polytopes:

DEFINITION 3.11. (**Degeneracy and Dimension of a Polytope**) Let $P \subset \mathcal{R}^d$ be a polytope. We say $P$ has dimension $q$ if the vector space spanned by its basis vectors has dimension $q$. We say $P$ is non-degenerate, or of full dimension, if $q = d$; otherwise we say $P$ is degenerate.

We note in passing that in case $P$ is degenerate and of dimension $q < d$, it is isomorphic to a non-degenerate polytope in the space $\mathcal{R}^q$.

As another building block for the subsequent work, we need the following:

THEOREM 3.12. (*Intersection of Polytope and Half Space*) *The intersection of a polytope with a half space is a polytope.*

PROOF. Without loss of generality, let us assume the half space is closed, i.e. it contains its boundary plane, and let us refer to those elements in $\mathcal{R}^d$ that do not belong to the half space as lying above the plane. Consider all vertices of the polytope $P$, and sort them based on whether they lie below, on, or above the plane of the half space of interest. If they all lie below or on the plane, the intersection is the original polytope. If they all lie above, the result is the empty set.

Now consider the case where some of the vertices lie on one side of the plane, and some on the other. For every pair of one vertex above and one vertex below the plane, consider the connection line between the points. By convexity, each of these connection lines lies fully inside $P$, and furthermore by construction, each intersects the plane. Let $\{p_i\}$ denote the set of all these intersection points. Now generate a new polytope formed by the vertices of $P$ and the set $\{p_i\}$. Since the $\{p_i\}$ lie inside $P$, the resulting polytope is indeed again $P$. Now consider the new polytope $\bar{P}$ formed from those vertices of $P$ below the plane and the set $\{p_i\}$. Apparently by construction all elements of $\bar{P}$ lie in the half space; and furthermore, every element in $P$ that is inside the half space lies in $\bar{P}$; so $\bar{P}$ is the intersection of $P$ with the half space. □

Now we address one of the important properties of polytopes, which sheds light on the relationship between polytopes and simplexes:

THEOREM 3.13. *(Polytope Decomposition) Any non-degenerate polytope in $\mathcal{R}^d$ can be written as a union of simplexes with disjoint interiors.*

PROOF. Pick one of the vertices $p_0$ of $P$. Consider all other vertices and connection lines from them to $p_0$. We call those connection lines that lie on the boundary of $P$ edges. Consider the space spanned by all vectors of edges emanating from $p_0$. The dimension of this space must be $d$, the dimension of the space we are considering. For if the dimension is lower than $d$, necessarily all other vertices of $P$ must also lie in this lower dimensional space, which is a contradiction to the non-degeneracy of $P$.

Of those edges spanning the full space, pick $d$ so that their span has full dimension. Consider their $d$ endpoints $p_1, ..., p_d$ opposite to $p_0$ and use them to define a plane in $\mathcal{R}^d$. Consider the intersection of $P$ with that half space defined by the plane and containing $p_0$. The result is a non-degenerate simplex $S$ with vertices $p_0$, $p_1, ... , p_d$. On the other hand, consider the intersection of $P$ with the half space defined by the plane and not containing $p_0$. The result is a polytope $P_1$ spanned by the all vertices of $P$ except $p_0$, i.e. with one less vertices than $P$. Furthermore we have by construction that $P_1 \cup S = Q$, while the intersection of $P_1$ and $S$ lies in the plane and is thus degenerate.

Now one proceeds inductively by removing vertices, until the resulting polytope is itself a simplex with $(d+1)$ vertices.                                         □

As consequences of the above arguments, we also obtain a result about intersections of polytopes:

THEOREM 3.14. *The intersection of two polytopes is a union of simplexes that intersect at most on their boundary.*

PROOF. Following de Morgan's laws, it is sufficient to consider the intersection of two simplexes. Furthermore, according to theorem 3.5, a simplex is the intersection of $(d+1)$ half spaces, so it is sufficient to consider intersections of simplexes and half spaces. According to the previous two theorems, these can be written as the union of almost disjoint simplexes.                                         □

Now we are ready to define the S-algebra of our interest:

DEFINITION 3.15. **(S-Algebra of Simplexes)** We define $S$ to be the S-algebra of finite unions of almost disjoint simplexes, and each element of $S$ is assigned a measure based on the sum of volumes of its defining simplexes.

Indeed the collection $S$ is an S-algebra, as the chain of theorems and propositions in this section shows. The most complicated part pertains to the intersection of two sets in $S$, which follows from theorem 3.14. In passing it is worthwhile to assert that the measure is indeed well-defined, and in particular that it is independent of the specific decomposition into simplexes that is chosen. However, to this end one has to merely observe that each element of $S$ has a unique representation through the smallest number of simplexes; and any larger representation splits some simplexes into other smaller simplexes. But elementary rules of linear algebra that carry into the Levi-Civita vector spaces assert that the measure of the original simplex equals the sum of measures of its constituent pieces.

It is particularly noteworthy to stress that the above construction is far from artificial; it is indeed the smallest S-algebra that is affine invariant, i.e. it is invariant under affine transformations in the sense of theorem 11.c.

THEOREM 3.16. *(Minimality of $S$) The $S$-algebra of simplexes $S$ is the smallest affine invariant $S$-algebra that contains the unit cube and assigns it unit measure.*

Before we come to the proof of the theorem, we need another observation:

LEMMA 3.17. *Every two points of the unit simplex have distance not exceeding $\sqrt{d}$.*

PROOF. Given two points $x,y$ in the unit simplex, we write them as $x = t_0 \cdot 0 + t_1 e_1 + ... + t_d e_d$ and $y = r_0 \cdot 0 + r_1 e_1 + ... + r_d e_d$; $\sum t_i = 1, \sum r_i = 1$, $t_i \geq 0$, $r_i \geq 0$. Let $dist(x, y)$ denote the Euclidean distance between $x$ and $y$, which because of the existence of roots of positive elements of the Levi-Civita field can be introduced and which has very similar properties. Then we have

$$dist(x,y) = \sqrt{(t_0 - r_0)^2 + ... + (t_d - r_d)^2} \leq \sqrt{(1-0)^2 + ... + (1-0)^2} \leq \sqrt{d}$$

$\square$

Now we are ready to proceed with the proof of the theorem:

PROOF. First, apparently the unit cube $U = [0,1]^d$ is contained in $S$, as it is the union of $d!$ closed simplexes, each of which is obtained by rotating and shifting the unit simplex defined in theorem 3.5. Since the $d!$ simplexes are almost disjoint and each has measure $1/d!$, we obtain that the measure of $U$ is indeed 1.

Let $S$ be an affine invariant $S$-algebra containing the unit cube. First, we show that $S$ contains the unit simplex. First we consider the non-trivial face of the unit simplex that has all coordinates non-vanishing, i.e. the face given by

$$H_0 = \{t_1 e_1 + ... + t_d e_d | \sum t_j = 1, t_j > 0\}.$$

Consider the point $c_0 = \frac{1}{d} e_1 + ... + \frac{1}{d} e_i + ... + \frac{1}{d} e_d = (\frac{1}{d}, \frac{1}{d}, ..., \frac{1}{d})$, which apparently lies in $H_0$ (in fact, geometrically it is the "center" of the face $H_0$). In the following, we will construct a sequence of affine transformations of the unit cube so that the resulting image fully contains the unit simplex and one of the images of its faces contains $H_0$. For the purpose of better illustration, we also show the sequence of transformation in figure 3.1 for the two-dimensional case.

First we observe that the cube $V = [-1, 1]^d$ is in the algebra $S$, since it apparently can be obtained as the image of $U$ under first the translation map $T_1$ given by $T_1(x) = x - (\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$ and followed by the stretching map $T_2$ given by $T_2(x) = 2x$, which shows that the cube $V = [-1, 1]^d$ is the image of an affine transformation of $U$, as shown in the second picture in 3.1 and thus in $S$.

Now we claim there is an affine transformation of $V$ such that the resulting image of $V$ contains the unit simplex and the image of one of the faces of $V$ contains $H_0$. To prove this claim, we will give an explicit construction of the affine transformation.

First, we perform a dilation of $V$ by $\sqrt{d}$, which is achieved by the stretching map $D(x) = \sqrt{d} \cdot x$. We call $A_1$ the image of $V$ under this transformation $D$; the result is shown in the third picture in 3.1. Next we perform an orthonormal transformation $R$ that takes the point $(\sqrt{d}, 0, ..., 0)$ in the center of the upper face of $D$ to $d \cdot c_0 = (1, 1, ..., 1)$, which apparently has the same length. To fully describe this affine transformation, we define its action on all Euclidean basis vectors $e_i$. We begin by demanding that $R(e_1) = \sqrt{d} \cdot c_0$. To achieve orthonormality, we apply the well-known Gram-Schmidt process to the set $\{\sqrt{d} \cdot c_0, e_2, ..., e_d\}$, which will give as

an orthogonal set of vectors $\{u_1, u_2, ..., u_d\}$ . More explicitly, using the notation $proj_u(v) = \frac{<u,v>}{<u,u>}u$, we inductively set

$u_1 = \sqrt{d} \cdot c_0$, $u_2 = e_2 - proj_{u_1}(e_2)$, $u_3 = e_3 - proj_{u_1}(e_3) - proj_{u_2}(e_3)$,..., $u_k = e_k - \sum_{j=1}^{k-1} proj_{u_j}(e_k)$. Here $< u, v >$ and $\|u\|$ are the inner product and norm defined as in any Euclidean space, and $proj_u(v) = \frac{<u,v>}{<u,u>}u$. So the desired affine transformation, $R$, is given by:

$$R(e_1) = \frac{u_1}{\|u_1\|} = \sqrt{d} \cdot c_0$$

$$R(e_2) = \frac{u_2}{\|u_2\|}, ..., R(e_d) = \frac{u_d}{\|u_d\|}$$

We call $A_2$ the image of $A_1$ under $R$; the result is shown in the fourth picture in 3.1. It is clear that $A_2$ is a cube of edge length $2\sqrt{d}$, we only have to show that one the faces of $A_2$ is parallel to $H_0$. In fact, the image of the face $F = \{(\sqrt{d}, ..., a_{d-1}, a_d) : -\sqrt{d} \le a_i \le \sqrt{d}\}$ is parallel to the plane containing $H_0$, since by construction they are both perpendicular to $c_0$, i.e. the inner product of $c_0$ with the difference of any two vectors in the image of $F$ or the difference of any two vectors in $H_0$ both vanish.

Finally, take the image of $A_2$ under the translation map $T$ given by $T(x) = x - (\frac{d+1}{d}, \frac{d+1}{d}, ..., \frac{d+1}{d})$. We denote the image of $A_2$ under $T$ by $A_3$, and the result is shown in the fifth picture in 3.1. By construction and the previous lemma, $A_3$ is an affine transformation of the original cube that contains the unit simplex, and the translation of the face $F$ contains $H_0$, because the face of the cube contains a disk of radius $\sqrt{d}$ centered at $c_0$, and by the preceding lemma, no point in $H_0$ is further from $c_0$ than $\sqrt{d}$.

Therefore, the unit simplex is the intersection of $A_3$ with $U$, as illustrated in the last picture of 3.1. Thus, $S$ contains the unit simplex.

Now we note that according to theorem 3.5, any simplex can be written as an affine image of the unit simplex. Since the S-algebra contains finite unions of its elements, we see that each element of $S$ is necessarily contained in the smallest S-algebra containing $U$. □

## 4. Measure theory in $\mathcal{R}^d$

We are now ready to use the S-algebras in the last section to introduce a measure on Levi-Civita vector spaces.

DEFINITION 4.1. (**S-Measure**) Let $A \subset \mathcal{R}^d$ be given. Then we say that $A$ is measurable under the S-algebra $S$ if for every $\epsilon > 0$ in $\mathcal{R}$, there exist sequences $(S_n)$ and $(T_n)$ of mutually disjoint open elements of $S$, such that $\cup_{n=1}^{\infty} S_n \subset A \subset \cup_{n=1}^{\infty} T_n$, $\sum_{n=1}^{\infty} m(S_n)$ and $\sum_{n=1}^{\infty} m(T_n)$ converge in $\mathcal{R}$, and $\sum_{n=1}^{\infty} m(T_n) - \sum_{n=1}^{\infty} m(S_n) \le \epsilon$.

As a special case, and as the most important case for the further discussion, we also define

DEFINITION 4.2. (**Measure**) We say a set $A \subset \mathcal{R}^d$ is measurable, or simplex measurable, if it is measurable under the S-algebra of simplexes.

The advantage of this approach is that it retains very close similarity with the earlier introduced measure [**32**], while being flexible enough to capture what
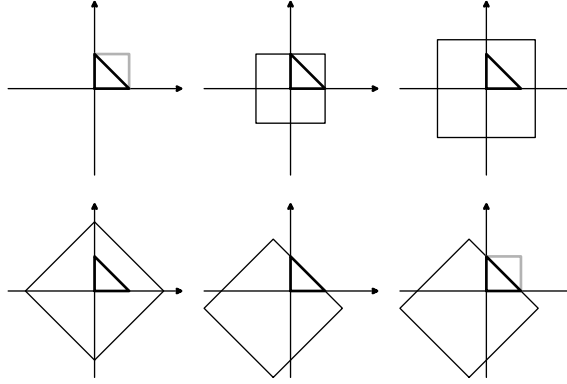
FIGURE 3.1. Top row: unit simplex and unit cube, stretching of cube and re-centering at origin, stretching cube by $\sqrt{d}$. Bottom row: rotating cube by $R$ , shifting cube, and intersecting with unit cube, resulting in the unit simplex.

is needed for the proof of the obtuse angle theorem. We now derive a few basic properties of the S-Measure.

PROPOSITION 4.3. *The unit cube $[0,1]^d \subset \mathcal{R}^d$ is simplex measurable with measure* 1. *Any countable set is measurable with measure* 0.

PROOF. The first statement follows directly from theorem 3.16. For the second statement, Let $(a_n)$ , $n = 1, 2, \dots$ denote the countable set and let $\epsilon > 0$ in $\mathcal{R}$ be given. We use that the cube is measurable, and consider affine images of the cube centered around the $a_n$, and we form sets $A_n$ as $A_n = \{a_n + \epsilon[-1, +1] \cdot \delta^n\} \cap \{\cup_{i=1}^{n-1} A_i\}$ where $\delta$ is a positive infinitely small number. Apparently the $A_n$ cover the set of interest and are mutually disjoint by construction. However, we have the sum of the measures $m(A_n)$ is bounded by $\epsilon \cdot \sqrt{\delta} < \epsilon$. Since $\epsilon$ is arbitrary, the result follows. □

One of the important results is the following:

PROPOSITION 4.4. *(Substitution Rule) If the set $A$ is simplex measurable and $\mathcal{A}$ is an affine transformation, then $\mathcal{A}(A)$ is measurable, and $m(\mathcal{A}(A)) = |det(\mathcal{A})| \cdot m(A)$.*

The proof follows directly from the definition of the measure in terms of elements of the S-algebra, the invariance of the S-algebra under affine transformation.

PROPOSITION 4.5. *(Countable Union of Measurable Sets) For each $k \in \mathbb{N}$, let $A_k \subset \mathcal{R}^d$ be measurable such that $(m(A_k))$ forms a null sequence. Then $\cup_{k=1}^{\infty} A_k$ is measurable and*

$$m\left(\cup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m\left(A_k\right).$$

*Moreover, if the sets $(A_k)_{k=1}^{\infty}$ are mutually disjoint, then*

$$m\left(\cup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m\left(A_k\right).$$

The proof of this proposition follows exactly as in the one dimensional case in $\mathcal{R}$ [**32**], it is simply necessary to replace all occurrences of intervals in the respective proof by elements of $S$. In a similar manner, one also obtains the proof of the following.

PROPOSITION 4.6. *Let $A, B \subset \mathcal{R}$ be measurable. Then $m(A \cup B) = m(A) + m(B) - m(A \cap B)$.*

## 5. The Obtuse Angle Theorem for Levi-Civita Vector Spaces

We now construct a proof of the Obtuse Angle Theorem for the vector spaces $\mathcal{R}^d$. The arguments follow very closely those of the real case [**8**], where in every step along the way it is confirmed that they hold in the corresponding necessary way in the Levi-Civita structures. As we will see, the most crucial part will turn out to be the properties of affine invariant measures discussed in the previous sections.

DEFINITION 5.1. **(Erdös Set)** Let $S_E$ be a set of finitely many points in $\mathcal{R}^d$. We call $S_E$ an Erdös Set if for any three points $s_0$, $s_1$ and $s_2$ , the angle formed by the points is non-obtuse.

As mentioned above, obtusity of an angle is well defined by virtue of the inner product. As it turns out, Erdös sets are closely related to so-called Klee sets, and their use helps in the subsequent proof:

DEFINITION 5.2. **(Klee Set)** Let $S_K$ be a set of finitely many points in $\mathcal{R}^d$. We call $S_K$ a Klee Set if for any two points $s_1$ and $s_2$ in the set, there is a "strip", i.e. a set bounded by two parallel hyperplanes, so that all points in $S_K$ are in the strip, and $s_1$ and $s_2$ are on opposite sides of the boundary.

For the purpose of practical comprehension, this property asserts that any two points are "on the outside" of the set and not merely in the span of the other points.

Let us now consider finite sets $S \subset \mathcal{R}^d$ of points, their convex hulls conv$(S)$, and general convex polytopes $Q \subset \mathcal{R}^d$. We assume without loss of generality that the set $S$ has full dimension, i.e. does not lie in a hyperplane; because otherwise it is possible to consider the problem merely in the space spanning the hyperplane, which is isomorphic to $\mathcal{R}^{d-1}$ by virtue of a rotation and translation, which preserves inner products and thus angles. We say that two sets touch iff they have at least one point in common, but have disjoint interior. For any set $Q \subset \mathcal{R}^d$ and any vector $s \in \mathcal{R}^d$, we denote by $Q + s$ the image of $Q$ under the translation that moves $0$ to $s$.

LEMMA 5.3. *We have the following relationship between the maximal cardinality of Erdös sets and Klee sets:*

$$2^d \leq \max(card(S_E)) \leq \max(card(S_K))$$

PROOF. The first inequality merely re-states that the vertex set of the unit cube in $\mathcal{R}^d$ is an Erdös set, as discussed above. The necessary arithmetic of the inner product and the non-negativity of all inner products of corner points follows exactly as in the real case. For the second inequality, let any two $s_i, s_j \in S_E$ be given. Define the hyperplane orthogonal to the line $[s_i, s_j]$ as $H_{ij} = \{x \in \mathcal{R}^d : x \cdot (s_i - s_j) = 0\}$. Then let the strip $S(i,j)$ be the regions between the hyperplanes $H_{ij} + s_i$ and $H_{ij} + s_j$, i.e. the translations of $H_{ij}$ by the vectors $s_i$ and $s_j$, respectively. Then it follows that any other point $s$ in $S_E$ lies inside the strip, since otherwise one of the

angles formed by the three points $\{s_i, s_j, s\}$ would be obtuse. So every Erdös set is a Klee set.     □

Now we introduce another type of finite point sets:

DEFINITION 5.4. **(Translating Convex Hull Set)** Let $P = \text{conv}(S_C)$ be the convex hull of the set $S_C$. We say that $S_C$ is a translating convex hull set if all translated convex hulls of the form

$$P - s_i$$

intersect in a common point, but they only touch.

LEMMA 5.5. *We then have*

$$2^d \le \max(card(S_K)) \le \max(card(S_C))$$

PROOF. We show that every Klee set $S_K$ is also a translating convex hull set. Let $S_K$ be a Klee set. We first observe that since for $s_i \in S_K$ we have $s_i \in P$, and thus $0 \in P - s_i$ for all $s_i$. Thus all convex hulls intersect (at the origin). Now, let two translating convex hull sets $P - s_i$ and $P - s_j$ be given. Consider the hyperplane through the origin $H_{ij}$ defined to be perpendicular to $(s_i - s_j)$. Then following the arguments in 5.3, the two sets lie on opposite sides of the hyperplane, so 0 is indeed their only common point.     □

We now relax the notion of the set $P$ in the above, and replace it with a general polytope $Q$ instead of $P$. We further replace the requirement of intersection in a common point by merely pairwise touching.

DEFINITION 5.6. **(Translating Polytope Set)** Let $Q$ be a general polytope. We say that $S_Q$ is a translating polytope set if all translated polytopes of the form

$$Q - s_i$$

pairwise touch.

LEMMA 5.7. *We then have*

$$2^d \le \max(card(S_K)) \le \max(card(S_C)) \le \max(card(S_Q))$$

PROOF. We merely observe every set $S_C$ is also a set $S_Q$.     □

The last step may appear somewhat strange in that we have vastly increased the number of sets we can consider, and yet as we shall see below, there will be no negative consequences in the remainder of our arguments.

In the next step we perform one of the most crucial transformation. We move from the general polytope $Q$ to another polytope $Q^*$ that satisfies the same properties, but has much nicer properties. That polytope is given through the following:

DEFINITION 5.8. **(Minkowski Symmetrization)** Let $Q$ be a polytope. Then we define the Minkowski Symmetrization $Q^*$ of $Q$ via

$$Q^* = \{\frac{1}{2}(x - y) : x, y \in Q\}$$

We note that since $Q^*$ is made of all differences of elements of $Q$, it is centrally symmetric, i.e. with $q$ it also contains $-q$. It is also easy to see that it is convex. Indeed, $Q^*$ is even again a polytope with vertices of the form $1/2(q_i - q_j)$, for vertices $q_i$, $q_j$ of $Q$, but this is immaterial for the subsequent arguments. In passing

we note that the special symmetric structure of $Q^*$ first introduced in [**21**] has various advantages and is used frequently in polytope theory. One of the interesting properties is the following:

LEMMA 5.9. *Let $Q$ be a polytope in $\mathcal{R}^d$, and $Q^*$ its Minkowski symmetrization. Then $Q + s_i$ and $Q + s_j$ touch if and only if $Q^* + s_i$ and $Q^* + s_j$ touch.*

PROOF. We first prove the weaker statement that $Q + s_i$ and $Q + s_j$ intersect if and only if $Q^* + s_i$ and $Q^* + s_j$ intersect. The proof follows from a sequence of relatively simple arithmetic steps. First note that $(Q^* + s_i) \bigcap (Q^* + s_j) \neq \emptyset$ is equivalent to the existence of $q'_i, q''_i, q'_j, q''_j \in Q$ such that $1/2(q'_i - q''_i) + s_i = 1/2(q'_j - q''_j) + s_j$. By bringing the double primed indices to the other side, this equation can also be written equivalently as

$$1/2(q'_i + q''_j) + s_i = 1/2(q'_j + q''_i) + s_j.$$

Because of the convexity of $Q$, we have that $q_i := 1/2(q'_i + q''_j)$ and $q_j := 1/2(q'_j + q''_i)$ lie in $Q$. So we see that the original condition implies the existence of $q_i$ and $q_j$ in $Q$ such that $q_i + s_i = q_j + s_j$, and so $Q + s_i$ and $Q + s_j$ intersect. But on the other hand, if the existence of such $q_i$ and $q_j$ in $Q$ is assumed, then writing $q_{i,j} = 1/2(q_{i,j} + q_{i,j})$, we also have shown the existence of $q'_i, q''_i, q'_j, q''_j \in Q$ such that $1/2(q'_i - q''_i) + s_i = 1/2(q'_j - q''_j) + s_j$, and hence $(Q^* + s_i) \bigcap (Q^* + s_j) \neq \emptyset$.

It remains to show the equivalence under touching. We first observe that two translates $Q + s_i$ and $Q + s_j$ touch if and only if they intersect, while $Q + s_i$ and $Q + s_j + \varepsilon(s_j - s_i)$ do not intersect for any $\epsilon > 0$. However, employing the just proved equivalence of intersection between $Q + s_i$ and $Q + s_j$ versus $Q^* + s_i$ and $Q^* + s_j$, we see that $Q + s_i$ and $Q + s_j + \varepsilon(s_j - s_i)$ not intersecting for any $\epsilon > 0$, $\epsilon \in \mathcal{R}$, is equivalent to $Q^* + s_i$ and $Q^* + s_j + \varepsilon(s_j - s_i)$ not intersecting for any such $\epsilon$. Thus we have proved the equivalence under touching. $\square$

Now we proceed to the final part in the chain of our estimates of cardinalities, in which we will obtain an upper bound on the maximal cardinality of the sets $S_Q$. We make use of many of the previous results of this paper, with the most central ingredients being the Minkowski symmetrization, the invariance of Levi-Civita measures under affine transformation, and the substitution rule for Levi-Civita measures.

LEMMA 5.10. **(Maximal Cardinality Theorem)** *All above sets have the same maximal cardinality; specifically, we have*

$$2^d = \max(card(S_E)) = \max(card(S_K)) = \max(card(S_C) = \max(card(S_Q)).$$

PROOF. Building on the previous lemmata of this section, we note that the only part necessary is to show that $\max(\operatorname{card}(S_Q)) \leq 2^d$. We begin with a more detailed study of a Minkowski symmetrized polytope that is translated by two different elements $s_i$ and $s_j$ of the set $S_Q$. Specifically, let us consider a point $x$ in the intersection of the translates, i.e. $x \in (Q^* + s_i) \bigcap (Q^* + s_j)$. We have $x - s_i \in Q^*$ and $x - s_j \in Q^*$, and since $Q^*$ is centrally symmetric, we also have $s_i - x = -(x - s_i) \in Q^*$. Further, since $Q^*$ is convex, we have that $1/2(s_i - s_j) = 1/2[(x - s_j) + (s_i - x)] \in Q^*$. Adding $s_j$, we see that $1/2(s_i + s_j)$ is contained in $Q^* + s_j$ for all $i$. Now let $P = conv(S_Q)$ denote the convex hull of the point set $S_Q$, and let us define the sets $P_j$ via

$$P_j = \frac{1}{2}(P + s_j).$$

Then we have that $P_j = \text{conv}\{1/2(s_i + s_j) : s_i \in S\} \subset Q^* + s_j$ , which implies that any two of the sets $P_j = 1/2(P + s_j)$ can only mutually touch. However, all sets $P_j$ are contained in $P$ . Because for any $x \in P_j$ there are positive $\lambda_i \in \mathcal{R}$ with $\sum_i \lambda_i = 1$ such that

$$x = \sum_i \lambda_i \cdot \frac{1}{2}(s_i + s_j) = \frac{1}{2}s_j + \frac{1}{2}\sum_i \lambda_i s_i.$$

But since $y = \sum \lambda_i s_i \in P$ by the definition of $P$, and because of the convexity of $P$, we have that $x = 1/2(s_j + y) \subset P$. So we have that $\cup_j P_j \subset P$, and since the $P_j$ are almost disjoint measurable sets, we have

$$\sum_j m(P_j) \leq m(P).$$

However, each of the $P_j$ is an affine image of the polytope $P$ under the transformation $M = 1/2 \cdot I + s_j$ where $I$ is the identity transformation, and so by the substitution rule of the Levi-Civita measure theory 4.4, we have

$$m(P_j) = \frac{1}{2^d} m(P).$$

Combining this with the previous inequality, we see that there can be at most $2^d$ different $P_j$, and hence there can be only at most $2^d$ different values of $s_j$. Thus $\max(\text{card}(S_Q)) \leq 2^d$, which completes our proof.    $\square$

As a direct consequence, the first equality of the previous Lemma entails our desired theorem:

THEOREM 5.11. *(Obtuse Angle Theorem for Levi-Civita Vector Spaces)* *Every set of more than* $2^d$ *points in the Levi-Civita vector space* $\mathcal{R}^d$ *admits at least one obtuse angle.*

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