

# SCARCITY OF FINITE ORBITS FOR RATIONAL FUNCTIONS OVER A NUMBER FIELD

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ABSTRACT. Let  $\phi$  be an endomorphism of degree  $d \geq 2$  of the projective line, defined over a number field  $K$ . Let  $S$  be a finite set of places of  $K$ , including the archimedean places, such that  $\phi$  has good reduction outside of  $S$ . The article presents two main results: the first result is a bound on the number of  $K$ -rational preperiodic points of  $\phi$  in terms of the cardinality of the set  $S$  and the degree  $d$  of the endomorphism  $\phi$ . This bound is quadratic in terms of  $d$  which is a significant improvement to all previous bounds on the number of preperiodic points in terms of the degree  $d$ . For the second result, if we assume that there is a  $K$ -rational periodic point of period at least two, then there exists a bound on the number of  $K$ -rational preperiodic points of  $\phi$  that is linear in terms of the degree  $d$ .

## 1. INTRODUCTION

In this article we prove the following theorem

**Theorem 1.** *Let  $K$  be a number field and  $S$  a finite set of places of  $K$  containing all the archimedean ones. Let  $\phi$  be an endomorphism of  $\mathbb{P}^1$ , defined over  $K$ , and  $d \geq 2$  the degree of  $\phi$ . Assume  $\phi$  has good reduction outside  $S$ . Then the number of  $K$ -rational preperiodic points is bounded by*

$$Q(|S|, d) = \alpha_1 d^2 + \beta_1 d + \gamma_1,$$

where  $\alpha_1$ ,  $\beta_1$  and  $\gamma_1$  are positive integers depending only on the cardinality of  $S$  and can be effectively computed.

In addition, if we assume that  $\phi$  has a  $K$ -rational periodic point of period at least two then the number of  $K$ -rational preperiodic points is bounded by

$$L(|S|, d) = \alpha_2 d + \beta_2,$$

where  $\alpha_2$  and  $\beta_2$  are positive integers depending only on the cardinality of  $S$  and can be effectively computed.

We emphasize that the constants  $\alpha_1, \alpha_2, \beta_1, \beta_2$  and  $\gamma_1$  in the theorem depend only on the cardinality of  $S$  (and thus implicitly on the degree  $[K : \mathbb{Q}]$ ) but not on the field  $K$  itself. An explicit definition of the bounds  $Q(|S|, d)$  and  $L(|S|, d)$  will be given in at the end Section 6.

Let  $\phi : \mathbb{P}_N \rightarrow \mathbb{P}_N$  be an endomorphism of degree  $d \geq 2$  defined over a number field  $K$ . Let  $\phi^n$  denote the  $n^{\text{th}}$  iterate of  $\phi$  under composition and  $\phi^0$  the identity map. The orbit of  $P \in \mathbb{P}_N(K)$  under  $\phi$  is the set  $O_\phi(P) = \{\phi^n(P) : n \geq 0\}$ . A point  $P \in \mathbb{P}_N(K)$  is called *periodic* under  $\phi$  if there is an integer  $n > 0$  such that  $\phi^n(P) = P$ ; the minimal such  $n$  is called the *period* of  $P$ . It is called *preperiodic* under  $\phi$  if there is an integer  $m \geq 0$  such that  $\phi^m(P)$  is periodic. A point that is preperiodic but not periodic is called a *tail* point. Let  $\text{Tail}(\phi, K)$ ,  $\text{Per}(\phi, K)$  and  $\text{PrePer}(\phi, K)$  denote the sets of  $K$ -rational tail, periodic and preperiodic points of  $\phi$ , respectively.

The set of preperiodic points in  $\mathbb{P}_N(\bar{K})$  of an endomorphism  $\phi : \mathbb{P}_N \rightarrow \mathbb{P}_N$  of degree  $d \geq 2$  defined over a number field  $K$ , where  $\bar{K}$  is the algebraic closure of  $K$ , is of bounded height (this is a special case of Northcott's theorem [Nor50]). Since a number field  $K$  possesses the Northcott property (*i.e.*, that every set of bounded height is finite, [BG06]), the set of  $K$ -rational preperiodic points of  $\phi$  is finite. In fact, from Northcott's proof, an explicit bound on  $\text{PrePer}(\phi, K)$  can be found in terms of the coefficients of  $\phi$ . The problem is to find a bound on the number of preperiodic points that depends in a "minimal" way on the map  $\phi$ . One of the main motivations for our research is the well known Morton and Silverman [MS94] conjecture which we state below

**Conjecture** (Uniform Boundedness Conjecture). *Let  $\phi : \mathbb{P}_N \rightarrow \mathbb{P}_N$  be an endomorphism of degree  $d \geq 2$  defined over a number field  $K$ . Let  $D$  be the degree of  $K$  over  $\mathbb{Q}$ . Then there exists a number  $C = C(D, N, d)$  such that  $\phi$  has at most  $C$  preperiodic points in  $\mathbb{P}^N(K)$ .*

This conjecture is an extremely strong uniformity conjecture. For example, the UBC (Uniform Boundedness Conjecture) for maps of degree 4 on  $\mathbb{P}_1$  defined over a number field  $K$  implies Merel's theorem (see [Mer96]), *i.e.* that the size of the torsion subgroup of an elliptic curve over a number field  $K$  is bounded only in terms of  $[K : \mathbb{Q}]$ . The conjecture can also be similarly applied to uniform boundedness of torsion subgroups of abelian varieties; for more details see [Fak01].

The Uniform Boundedness Conjecture seems extremely difficult to prove even in the simplest case when  $(K, N, d) = (\mathbb{Q}, 1, 2)$ . Furthermore, in this special case, explicit conjectures have been formulated. For instance, B. Poonen [Poo98] conjectured an explicit bound when  $\phi$  is a quadratic polynomial map over  $\mathbb{Q}$ . Since every such quadratic polynomial map is conjugate to a polynomial of the form  $\phi_c(z) = z^2 + c$  with  $c \in \mathbb{Q}$  we can state Poonen's conjecture as follows: Let  $\phi_c \in \mathbb{Q}[z]$  be a polynomial of degree 2 of the form  $\phi_c(z) = z^2 + c$  with  $c \in \mathbb{Q}$ . Then  $|\text{PrePer}(\phi_c, \mathbb{Q})| \leq 9$ . B. Hutz and P. Ingram [HI13] have shown that Poonen's conjecture holds when the numerator and denominator of  $c$  don't exceed  $10^8$ .

A natural relaxation of the uniform boundedness conjecture is to restrict our study to families of rational functions given in terms of good reduction. A rational map  $\phi : \mathbb{P}_1 \rightarrow \mathbb{P}_1$  of degree  $d \geq 2$  defined over a number field  $K$  is said to have *good reduction* at a non zero prime  $\mathfrak{p}$  of  $K$  if  $\phi$  can be written as  $\phi = [F(X, Y) : G(X, Y)]$  where  $F, G \in R_{\mathfrak{p}}[X, Y]$  are homogeneous polynomials of degree  $d$ , such that the resultant of  $F$  and  $G$  is a  $\mathfrak{p}$ -unit, where  $R_{\mathfrak{p}}$  is the localization of the ring of integers of  $K$  at  $\mathfrak{p}$ . The map  $\phi$  is said to have *bad reduction* at a prime  $\mathfrak{p}$  of  $K$  if  $\phi$  does not have good reduction at  $\mathfrak{p}$ . For a fixed finite set  $S$  of places of  $K$  containing all the archimedean ones, we say that  $\phi$  has good reduction outside of  $S$  if it has good reduction at each place  $\mathfrak{p} \notin S$ .

In the special case of rational functions  $\phi : \mathbb{P}_1 \rightarrow \mathbb{P}_1$ , there are several results giving a uniform bound on the number of periodic/preperiodic points of  $\phi$  depending on the cardinality of a finite set of places  $S$ , which includes all archimedean places, together with the constants  $[K : \mathbb{Q}]$  and  $\deg(\phi)$ , under the assumption that  $\phi$  has good reduction outside of  $S$  (*e.g.*, [Nar89, MS94, Ben07, Can07, Can10, CP16, CV, Tro17]).

We recall the definition of the  $\mathfrak{p}$ -adic logarithmic distance on  $\mathbb{P}_1(K)$  for a finite place  $\mathfrak{p}$  of a number field  $K$ : Let  $P_1 = [x_1 : y_1]$  and  $P_2 = [x_2 : y_2]$  be points in  $\mathbb{P}_1(K)$ . We will denote by

$$\delta_{\mathfrak{p}}(P_1, P_2) = v_{\mathfrak{p}}(x_1 y_2 - x_2 y_1) - \min\{v_{\mathfrak{p}}(x_1), v_{\mathfrak{p}}(y_1)\} - \min\{v_{\mathfrak{p}}(x_2), v_{\mathfrak{p}}(y_2)\}$$

the  $\mathfrak{p}$ -adic logarithmic distance between the points  $P_1$  and  $P_2$ .

In [CV] the first and third author proved a bound on the number of periodic points of a rational function  $\phi$  that is *linear* in the degree of  $\phi$ , but exponential in  $|S|$ . Roughly, the number of periodic points is bounded by  $2^{2^5|S|}d + 2^{2^{77}|S|}$ . To prove this result, the authors used the following lemma.

**Lemma 1.1** (Four-Point Lemma A [CV]). *Let  $\phi$  be an endomorphism of  $\mathbb{P}_1$  of degree  $d \geq 2$ , defined over a number field  $K$ . Let  $S$  be a finite set of places of  $K$  containing all the archimedean ones, such that  $\phi$  has good reduction outside  $S$ . Let  $A, B, C, D \in \mathbb{P}_1(K)$  be four distinct points such that also the images  $\phi(A), \phi(B), \phi(C), \phi(D)$  are distinct. Let  $\mathcal{P}$  be the set of points  $P \in \mathbb{P}_1(K)$  satisfying the following four equations for all  $\mathfrak{p} \notin S$ .*

$$(1.1) \quad \begin{aligned} \delta_{\mathfrak{p}}(A, P) &= \delta_{\mathfrak{p}}(\phi(A), \phi(P)), & \delta_{\mathfrak{p}}(B, P) &= \delta_{\mathfrak{p}}(\phi(B), \phi(P)), \\ \delta_{\mathfrak{p}}(C, P) &= \delta_{\mathfrak{p}}(\phi(C), \phi(P)), & \delta_{\mathfrak{p}}(D, P) &= \delta_{\mathfrak{p}}(\phi(D), \phi(P)). \end{aligned}$$

Then  $\mathcal{P}$  is finite and

$$(1.2) \quad |\mathcal{P}| \leq 2^{2^5|S|}d + 2^{2^{77}|S|}.$$

One obtains a bound on the number of periodic points by proving that the set of periodic points is contained in the set  $\mathcal{P}$  in Four-Point Lemma A. The lemma is directly related to the Siegel-Mahler theorem (cf. [FZ14]) and Evertse's explicit bound on the number of solutions of the  $S$ -unit equation (see [Eve84]), that can be combined and presented in the following way.

**Lemma 1.2** (Three-Point Lemma A). *Let  $K$  be a number field and  $S$  a finite set of places of  $K$  containing all the archimedean ones. Let  $A, B, C \in \mathbb{P}_1(K)$  be three distinct points. Let  $\mathcal{P}$  be the set of points  $P \in \mathbb{P}_1(K)$  satisfying the following three equations for all  $\mathfrak{p} \notin S$ .*

$$(1.3) \quad \delta_{\mathfrak{p}}(A, P) = 0, \quad \delta_{\mathfrak{p}}(B, P) = 0, \quad \delta_{\mathfrak{p}}(C, P) = 0$$

Then  $\mathcal{P}$  is finite and

$$(1.4) \quad |\mathcal{P}| \leq 3 \cdot 7^{4|S|}.$$

The second author proved in [Tro17] an inverse relationship between periodic points and tail points. Using Three-Point Lemma A, and combining with results from [CV], we can improve the result in [Tro17]. Before stating the result we recall some definitions: A point  $[a : b] \in \mathbb{P}_1(K)$  is *critical* for a rational function  $\phi: \mathbb{P}_1 \rightarrow \mathbb{P}_1$  if the order of zero at  $[a : b]$  of the algebraic condition  $\phi([x : y]) = \phi([a : b])$  is greater than 1 (if the point  $P$  and its image  $\phi(P)$  are finite, then  $P$  is critical if and only if  $\phi'(P) = 0$ ). A point *belongs to a critical cycle* of  $\phi$  if it belongs to the orbit of a critical periodic point. We denote by  $\text{Per}_0(\phi, K)$  the set of  $K$ -rational (periodic) points that belong to some critical cycle of  $\phi$ .

**Theorem 2.** *Let  $\phi: \mathbb{P}_1 \rightarrow \mathbb{P}_1$  be a rational map of degree  $d \geq 2$  defined over a number field  $K$ . Suppose  $\phi$  has good reduction outside a finite set of places  $S$ , including all archimedean ones.*

- (1) *If there exist three  $K$ -rational points such that each point is either a tail point or belongs to a critical cycle of  $\phi$  then*

$$|\text{Per}(\phi, K)| \leq 3 \cdot 7^{4|S|} + 3.$$

- (2) *If there exist at least four  $K$ -rational periodic points of  $\phi$  then*

$$|\text{Tail}(\phi, K)| + |\text{Per}_0(\phi, K)| \leq 12 \cdot 7^{4|S|}.$$

It is important to remark that the bounds in the theorem are *independent* of the degree  $d$  of the map  $\phi$ . Furthermore, the hypothesis of the second statement in the result is optimal in the sense that it is impossible to get any result of this form (i.e. independent of  $d$ ) when  $|\text{Per}(\phi, K)| < 4$ , as shown by the examples in [Tro17, §5].

The main goal of this article is to combine the results from [CV] and [Tro17] as well as techniques from Canci and Paladino [CP16], to obtain a bound on the number of preperiodic points which is better than previous results in terms of the degree  $d$  of  $\phi$ . In fact if the rational map has a  $K$ -rational periodic point of period at least 2 then we prove a bound on the number of preperiodic points of a rational function that is *linear* in the degree of the rational function, but exponential in  $|S|$ . In any other case, we provide a bound on the number of preperiodic points of a rational function that is *quadratic* in the degree of the rational function, but exponential in  $|S|$ .

Note that Benedetto's bound [Ben07] on the number of preperiodic points of polynomial maps is  $O(d^2/\log d)$  (for *most* cases of polynomial maps) has been the best estimate so far (and only for polynomials!).

In the process of proving this theorem, we prove and use the following “n-point lemmas”, which are generalizations of Three-Point Lemma A. The benefit of these lemmas over Four-Point Lemma A above is that they are independent of the map  $\phi$ , and therefore much more useful.

**Lemma 1.3** (Three-Point Lemma B). *Let  $K$  be a number field and  $S$  a finite set of places of  $K$  containing all the archimedean ones. Let  $Q_1, Q_2, Q_3$  be three distinct points in  $\mathbb{P}_1(K)$ . Given a fixed choice of nonnegative integers  $n_{i,\mathfrak{p}}$  for each  $i \in \{1, 2, 3\}$  and each  $\mathfrak{p} \notin S$ , the set*

$$(1.5) \quad \{P \in \mathbb{P}_1(K) \mid \delta_{\mathfrak{p}}(P, Q_i) = n_{i,\mathfrak{p}} \text{ for each } i \in \{1, 2, 3\} \text{ and } \mathfrak{p} \notin S\}$$

*has cardinality bounded by a number  $B(|S|)$ , depending only on  $|S|$  (see Section 6 for an explicit choice of  $B(|S|)$ ).*

**Lemma 1.4** (Four-Point Lemma B). *Let  $K$  be a number field and  $S$  a finite set of places of  $K$  containing all the archimedean ones. Let  $Q_1, Q_2, Q_3, Q_4$  be four distinct points in  $\mathbb{P}_1(K)$ . The set*

$$\mathcal{F} = \{P \in \mathbb{P}_1(K) \mid \delta_{\mathfrak{p}}(P, Q_1) = \delta_{\mathfrak{p}}(P, Q_2), \delta_{\mathfrak{p}}(P, Q_3) = \delta_{\mathfrak{p}}(P, Q_4), \forall \mathfrak{p} \notin S\}$$

*is finite with cardinality bounded by a number  $C(3, |S|) + 2$ , depending only on  $|S|$  (see Section 6 for an explicit choice of  $C(3, |S|)$ ).*

**Lemma 1.5** (Three-Point Lemma C). *Let  $K$  be a number field and  $S$  a finite set of places of  $K$  containing all the archimedean ones. Let  $Q_1, Q_2, Q_3$  be three distinct points in  $\mathbb{P}_1(K)$ . The set*

$$\mathcal{T} = \{P \in \mathbb{P}_1(K) \mid \delta_{\mathfrak{p}}(P, Q_1) = \delta_{\mathfrak{p}}(P, Q_2) = \delta_{\mathfrak{p}}(P, Q_3), \mathfrak{p} \notin S\}$$

*is finite with cardinality bounded by a number  $B(|S|)$  (same bound as in Three-Point Lemma B).*

## 2. PRELIMINARIES

In the present article we will use the following notation:

**Notation 2.1.**  $K$  a number field;

$\bar{K}$  an algebraic closure of  $K$ ;

$R$  the ring of integers of  $K$ ;

$\mathfrak{p}$  a non-zero prime ideal of  $R$ ;

$v_{\mathfrak{p}}$  the  $\mathfrak{p}$ -adic valuation on  $K$  corresponding to the prime ideal  $\mathfrak{p}$  (we always assume

$v_{\mathfrak{p}}$  to be normalized so that  $v_{\mathfrak{p}}(K^*) = \mathbb{Z}$ );

$S$  a fixed finite set of places of  $K$  including all archimedean places;

$R_S = \{x \in K : v_{\mathfrak{p}}(x) \geq 0 \text{ for every prime ideal } \mathfrak{p} \notin S\}$  the ring of  $S$ -integers;

$R_S^* = \{x \in K : v_{\mathfrak{p}}(x) = 0 \text{ for every prime ideal } \mathfrak{p} \notin S\}$  the group of  $S$ -units;

$\text{Per}(\phi, K)$  the set of  $K$ -rational periodic points of  $\phi$ ;

$\text{Tail}(\phi, K)$  the set of  $K$ -rational tail points of  $\phi$ ;

$\text{PrePer}(\phi, K)$  the set of  $K$ -rational preperiodic points of  $\phi$ ;

$\text{Tail}(\phi, K, P)$  the set of  $K$ -rational tail points of  $P$  with respect to  $\phi$ .

$\text{Per}_0(\phi, K)$  the set of  $K$ -rational periodic points belonging to some critical cycle of  $\phi$ .

We start this section by recalling the definition of the  $\mathfrak{p}$ -adic logarithmic distance on  $\mathbb{P}_1(K)$  for a finite place  $\mathfrak{p}$  in a number field  $K$ .

**Definition 2.2.** Let  $P_1 = [x_1 : y_1]$  and  $P_2 = [x_2 : y_2]$  be points in  $\mathbb{P}_1(K)$ . We will denote by

$$\delta_{\mathfrak{p}}(P_1, P_2) = v_{\mathfrak{p}}(x_1 y_2 - x_2 y_1) - \min\{v_{\mathfrak{p}}(x_1), v_{\mathfrak{p}}(y_1)\} - \min\{v_{\mathfrak{p}}(x_2), v_{\mathfrak{p}}(y_2)\}$$

the  $\mathfrak{p}$ -adic logarithmic distance on  $\mathbb{P}_1(K)$  between the points  $P_1$  and  $P_2$ .

Note that  $\delta_{\mathfrak{p}}(P_1, P_2)$  is independent of the choice of homogeneous coordinates and  $\delta_{\mathfrak{p}}(P_1, P_2)$  is 0 if and only if the points  $P_1$  and  $P_2$  are distinct modulo  $\mathfrak{p}$ .

The following definition introduces the idea of normalized forms with respect to  $\mathfrak{p}$ .

**Definition 2.3.**

- (1) We say that  $P = [x : y] \in \mathbb{P}_1(K)$  is in normalized form with respect to  $\mathfrak{p}$  if

$$\min\{v_{\mathfrak{p}}(x), v_{\mathfrak{p}}(y)\} = 0.$$

- (2) Let  $\phi$  be an endomorphism of  $\mathbb{P}_1$ , defined over  $K$ . Assume  $\phi$  is given by

$$\phi = [F(X, Y) : G(X, Y)]$$

where  $F, G \in K[X, Y]$  are homogeneous polynomials with no common factors. We say that the pair  $(F, G)$  is normalized with respect to  $\mathfrak{p}$  or that  $\phi$  is in normalized form with respect to  $\mathfrak{p}$  if  $F, G \in R_{\mathfrak{p}}[X, Y]$  and at least one coefficient of  $F$  or  $G$  is not in the maximal ideal of  $R_{\mathfrak{p}}$ . Equivalently,  $\phi = [F : G]$  is normalized with respect to  $\mathfrak{p}$  if

$$F(X, Y) = a_0 X^d + a_1 X^{d-1} Y + \dots + a_{d-1} X Y^{d-1} + a_d Y^d$$

and

$$G(X, Y) = b_0 X^d + b_1 X^{d-1} Y + \dots + b_{d-1} X Y^{d-1} + b_d Y^d$$

satisfy

$$\min\{v_{\mathfrak{p}}(a_0), \dots, v_{\mathfrak{p}}(a_d), v_{\mathfrak{p}}(b_0), \dots, v_{\mathfrak{p}}(b_d)\} = 0.$$

**Remark 2.4.** Note that if  $P = [x_1 : x_2]$  and  $Q = [y_1 : y_2]$  are in normalized form with respect to  $\mathfrak{p}$  then  $\delta_{\mathfrak{p}}(P, Q) = v_{\mathfrak{p}}(x_1 y_2 - x_2 y_1)$ .

Since  $R_{\mathfrak{p}}$  is a discrete valuation ring, we can always find a representation of  $P$  and  $\phi$  in normalized form with respect to  $\mathfrak{p}$ . However, it is not always true that the same representation is normalized for every  $\mathfrak{p}$ .

**Definition 2.5.** Consider  $P \in \mathbb{P}_1(K)$  and write  $P = [a : b]$  with  $a, b \in R_S$ . We say that  $[a : b]$  are *S-coprime coordinates* for  $P$  if  $\min\{v_{\mathfrak{p}}(a), v_{\mathfrak{p}}(b)\} = 0$  for every prime  $\mathfrak{p} \notin S$ .

Even though the definition of good reduction was given in the introduction, we recall the definition below for convenience of the reader.

**Definition 2.6.** Let  $\phi$  be an endomorphism of  $\mathbb{P}_1$ , defined over  $K$  and write  $\phi = [F : G]$  in normalized form with respect to  $\mathfrak{p}$ . We say that  $\phi$  has *good reduction* at  $\mathfrak{p}$  if  $\tilde{F}(X, Y) = \tilde{G}(X, Y) = 0$  has no solutions in  $\mathbb{P}_1(\bar{k})$ , where  $\tilde{F}$  and  $\tilde{G}$  are the reductions of  $F$  and  $G$  modulo  $\mathfrak{p}$  respectively and  $k$  is the residue field of  $R_{\mathfrak{p}}$ . We say that  $\phi$  has *good reduction outside S* if  $\phi$  has good reduction at  $\mathfrak{p}$  for every  $\mathfrak{p} \notin S$ .

Let  $\phi$  be an endomorphism of  $\mathbb{P}^1$ , defined over  $K$  and  $P \in \mathbb{P}_1(K)$ . We say that a point  $Q \in \mathbb{P}^1(K)$  is in the *tail of P* if  $P$  is in the orbit of  $Q$  under  $\phi$ . We denote  $\text{Tail}(\phi, K, P)$  the set of  $K$ -rational tail points of  $P$  with respect to  $\phi$ . Notice that a point  $R \in \mathbb{P}_1(K)$  is a tail point if it is non-periodic and in the tail of some periodic point. We define the *tail length* of a tail point  $R$  as the minimal natural number  $n$  such that  $\phi^n(P)$  is periodic.

Between the many interesting properties of the  $\mathfrak{p}$ -adic logarithmic distance we state the following that will be relevant for our work.

**Proposition 2.7.** [MS95, Proposition 5.1] *For all  $P_1, P_2, P_3 \in \mathbb{P}_1(K)$ , we have*

$$(2.1) \quad \delta_{\mathfrak{p}}(P_1, P_3) \geq \min\{\delta_{\mathfrak{p}}(P_1, P_2), \delta_{\mathfrak{p}}(P_2, P_3)\}.$$

**Proposition 2.8.** [MS95, Proposition 5.2] *Let  $\phi$  be an endomorphism of  $\mathbb{P}_1$  defined over  $K$  with good reduction at  $\mathfrak{p}$ . Then for any  $P, Q \in \mathbb{P}(K)$  we have*

$$(2.2) \quad \delta_{\mathfrak{p}}(\phi(P), \phi(Q)) \geq \delta_{\mathfrak{p}}(P, Q).$$

A direct application of the previous two propositions was deduced by Canci and Paladino [CP16].

**Lemma 2.9.** [CP16, Lemma 4.1] *Let  $\phi$  be an endomorphism of  $\mathbb{P}_1$  defined over  $K$  with good reduction at  $\mathfrak{p}$ . Let  $P_0 \in \mathbb{P}_1(K)$  be a fixed point of  $\phi$ . Let  $a, b$  be integers with  $0 < a < b$  and  $P_b, P_a \in \mathbb{P}_1(K)$  such that  $\phi^b(P_b) = \phi^a(P_a) = P_0$  and  $\phi^{b-a}(P_b) = P_a$ . Then*

$$(2.3) \quad \delta_{\mathfrak{p}}(P_b, P_a) = \delta_{\mathfrak{p}}(P_b, P_0) \leq \delta_{\mathfrak{p}}(P_a, P_0).$$

### 3. DYNAMICAL PROPERTIES OF THE LOGARITHMIC DISTANCE

As mentioned in the introduction, the second author proved in [Tro17] a strong arithmetic relation between  $K$ -rational tail points and  $K$ -rational periodic points. We state this arithmetic relation below:

**Proposition 3.1.** [Tro17, Corollary 2.23] *Let  $\phi$  be an endomorphism of  $\mathbb{P}_1$ , defined over  $K$ . Suppose  $\phi$  has good reduction outside  $S$ . Let  $R \in \mathbb{P}_1(K)$  be a tail point and let  $n$  be the period of the periodic part of the orbit of  $R$ . Let  $P \in \mathbb{P}_1(K)$  be any periodic point that is not  $\phi^{mn}(R)$  for some  $m$ . Then  $\forall \mathfrak{p} \notin S \quad \delta_{\mathfrak{p}}(P, R) = 0$ .*

In a similar vein, the first and third author proved the following proposition relating periodic points and points belonging to a critical cycle.

**Proposition 3.2.** [CV, Corollary 2.6] *Let  $\phi$  be an endomorphism of  $\mathbb{P}_1$ , defined over  $K$ . Suppose  $\phi$  has good reduction outside  $S$ . Let  $P \in \mathbb{P}_1(K)$  be a periodic point and let  $Q \in \mathbb{P}_1(K)$  belong to a critical cycle. Then  $\forall \mathfrak{p} \notin S \quad \delta_{\mathfrak{p}}(P, Q) = 0$ .*

We show how the two propositions together with Three-Point Lemma A can be used to prove Theorem 2.

*Proof of Theorem 2.*

- (1) Assume there exist three  $K$ -rational points  $R_1, R_2, R_3$  such each point is either a tail point or belongs to a critical cycle of  $\phi$ . Then by Propositions 3.1 and 3.2 we get that for any periodic point  $P \in \text{Per}(\phi, K) \setminus \{R_1, R_2, R_3\}$  we have

$$\forall \mathfrak{p} \notin S, \quad \forall 1 \leq i \leq 3, \quad \delta_{\mathfrak{p}}(P, R_i) = 0.$$

We can therefore apply Three-Point Lemma A to obtain the required bound.

- (2) Assume there exist four  $K$ -rational periodic points of  $\phi$  which we denote by  $P_1, P_2, P_3$  and  $P_4$ . Let  $R \in \text{Tail}(\phi, K) \cup \text{Per}_0(\phi, K)$ . If  $R$  is a tail point there can exist at most one periodic point  $P$  such that  $\phi^{mn}(R) = P$  for some  $m$ , where  $n$  is the period of  $P$ ; therefore at least three of  $P_1, P_2, P_3, P_4$  do not satisfy this property, and we can assume without loss of generality that  $P_1, P_2, P_3$  do not satisfy this property. If  $R$  is a periodic point, we can assume again without loss of generality that it is distinct from  $P_1, P_2, P_3$ . By Propositions 3.1 and 3.2 we get that

$$\forall \mathfrak{p} \notin S, \quad \forall 1 \leq i \leq 3, \quad \delta_{\mathfrak{p}}(P_i, R) = 0.$$

We can therefore apply Three-Point Lemma A (four times, depending on which three points of  $P_1, P_2, P_3, P_4$  we choose) to obtain the required bound. □

#### 4. THE $n$ -POINT LEMMAS

We would like to write every point in  $S$ -coprime coordinates. However,  $R_S$  is generally not a principal ideal domain and thus there exist points in  $\mathbb{P}_1(K)$  that do not have  $S$ -coprime coordinates. To avoid this problem we use the same argument as in [Can07]. For the reader's convenience we write this argument below.

Let  $\mathbf{a}_1, \dots, \mathbf{a}_h$  be a full system of integral representatives for the ideal classes of  $R_S$ . Hence, for each  $i \in \{1, \dots, h\}$  there is an  $S$ -integer  $\alpha_i \in R_S$  such that

$$\mathbf{a}_i^h = \alpha_i R_S.$$

Let  $L$  be the extension of  $K$  given by

$$L = K(\zeta, \sqrt[h]{\alpha_1}, \dots, \sqrt[h]{\alpha_h})$$

where  $\zeta$  is a primitive  $h$ -th root of unity and  $\sqrt[h]{\alpha_i}$  is a chosen  $h$ -th root of  $\alpha_i$ .

We denote by  $\sqrt{R_S^*}$ ,  $\sqrt{R_S}$  and  $\sqrt{K}$  the radicals in  $L^*$  of  $R_S^*$ ,  $R_S$  and  $K$  respectively. Denote by  $\mathbb{S}$  the set of places of  $L$  lying above the places in  $S$  and by  $R_{\mathbb{S}}$  and  $R_{\mathbb{S}}^*$  the ring of  $\mathbb{S}$ -integers and the group of  $\mathbb{S}$ -units, respectively in  $L$ . By definition  $R_S^* \cap \sqrt{K^*} = \sqrt{R_S^*}$  and  $\sqrt{R_S^*}$  is a subgroup of  $L^*$  of free rank  $s - 1$  by Dirichlet's unit theorem.

**Lemma 4.1.** *Assume the notation above. There exist fixed representations  $[x_P : y_P] \in \mathbb{P}_1(L)$  for every rational point  $P \in \mathbb{P}^1(K)$  satisfying the following two conditions.*

(a) For every  $P \in \mathbb{P}_1(K)$ , we have  $x_P, y_P \in \sqrt{K^*}$  and

$$x_P R_{\mathbb{S}} + y_P R_{\mathbb{S}} = R_{\mathbb{S}}.$$

(b) If  $P, Q \in \mathbb{P}^1(K)$  then

$$x_P y_Q - y_P x_Q \in \sqrt{K^*}.$$

*Proof.* Let  $P = [x : y]$  be a representation of  $P$  in  $\mathbb{P}_1(K)$  and consider  $\mathbf{b} \in \{\mathbf{a}_1, \dots, \mathbf{a}_h\}$  a representative of  $xR_S + yR_S$ . We can find  $\beta \in K^*$  such that  $\mathbf{b}^h = \beta R_S$ . Then there is  $\lambda \in K^*$  such that

$$(4.1) \quad (xR_S + yR_S)^h = \lambda^h \beta R_S.$$

We define in  $L$

$$x' = \frac{x}{\lambda^h \sqrt[h]{\beta}} \quad y' = \frac{y}{\lambda^h \sqrt[h]{\beta}}$$

and with this definition, it is clear that  $x', y' \in \sqrt{K^*}$  such that  $x'R_S + y'R_S = R_S$ . Furthermore, let  $P = [x'_1, y'_1]$  and  $Q = [x'_2 : y'_2]$  where

$$x'_i = \frac{x_i}{\lambda_i^h \sqrt[h]{\beta_i}} \quad y'_i = \frac{y_i}{\lambda_i^h \sqrt[h]{\beta_i}}$$

and  $\lambda_i, \beta_i$  are as the ones described in equation (4.1) for  $i \in \{1, 2\}$ . Then

$$(x'_1 y'_2 - y'_1 x'_2)^h = \frac{(x_1 y_2 - y_1 x_2)^h}{\lambda_1^h \lambda_2^h \beta_1 \beta_2} \in K^*.$$

□

**Definition 4.2.** A point  $P \in \mathbb{P}_1(K)$  written as in Lemma 4.1 is said to be written in  $S$ -radical coprime coordinates.

For the rest of this section we assume the above notation for  $L, \mathbb{S}, \sqrt{K}, \sqrt{R_S^*}$ , and  $\sqrt{R_S}$ .

Before we prove Three-Point Lemma B, Four-Point Lemma B and Three-Point Lemma C, we need to recall some results on the  $S$ -unit equation. Below we cite a result from Beukers and Schlickewei that gives a bound on the number of solutions of the  $S$ -unit equation in two variables where these solutions lie in a multiplicative subgroup.

**Theorem 3** (Beukers and Schlickewei [BS96]). *Let  $K$  be a number field and  $\Gamma$  be a subgroup of  $(K^*)^2 = K^* \times K^*$  of rank  $r$ . Then the equation*

$$x + y = 1$$

*has at most  $2^{8(r+1)}$  solutions with  $(x, y) \in \Gamma$ .*

**Corollary 1.** *Let  $K$  be a number field and  $\Gamma_0$  be a subgroup of  $K^*$  of rank  $r$ . Consider  $\Gamma = \Gamma_0 \times \Gamma_0$  and assume  $a, b \in K^*$ . Then the equation*

$$ax + by = 1$$

*has at most  $2^{8(2r+2)}$  solutions with  $(x, y) \in \Gamma$ .*

Next we quote a result from Evertse, Schlickewei and Schmidt that gives a bound on the number of solutions of the  $S$ -unit equation in three or more variables where these solutions lie in a multiplicative subgroup.

**Theorem 4** (Evertse, Schlickewei and Schmidt [ESS02]). *Let  $K$  be a number field and  $\Gamma$  be a subgroup of  $(K^*)^n$  of rank  $r$ . Assume  $a_1, \dots, a_n \in (K^*)^n$ . Then the equation*

$$a_1x_1 + \dots + a_nx_n = 1$$

*has at most  $e^{(6n)^{3n}(r+1)}$  solutions with  $(x_1, \dots, x_n) \in \Gamma$  and  $\sum_{i \in I} a_i x_i \neq 0$  for every nonempty subset  $I$  of  $\{1, \dots, n\}$ .*

In general, we will use Corollary 1 and Theorem 4 with  $\Gamma = \sqrt{R_S^*} \times \sqrt{R_S^*}$  and  $\Gamma = (\sqrt{R_S^*})^n$  respectively. The bounds in each case will be  $2^{8(2|S|)}$  and  $e^{(6n)^{3n}(n|S|+1-n)}$  and they will be denoted by  $B(|S|)$  and  $C(n, |S|)$ , respectively. In what follows we will prove Three-Point Lemma B, Four-Point Lemma B and Three-Point Lemma C.

**Lemma 4.3** (Three-Point Lemma B). *Let  $K$  be a number field and  $S$  a finite set of places of  $K$  containing all the archimedean ones. Let  $Q_1, Q_2, Q_3$  be three distinct points in  $\mathbb{P}_1(K)$ . Given a fixed choice of nonnegative integers  $n_{i,\mathfrak{p}}$  for each  $i \in \{1, 2, 3\}$  and each  $\mathfrak{p} \notin S$ , the set*

$$(4.2) \quad \{P \in \mathbb{P}_1(K) \mid \delta_{\mathfrak{p}}(P, Q_i) = n_{i,\mathfrak{p}} \text{ for each } i \in \{1, 2, 3\} \text{ and } \mathfrak{p} \notin S\}$$

*has cardinality bounded by  $B(|S|)$ .*

*Proof.* The set in (4.2) is empty if the set of nonzero numbers  $n_{i,\mathfrak{p}}$  is infinite. Otherwise there are three elements  $C_1, C_2, C_3 \in \sqrt{K^*}$  such that

$$v_{\mathfrak{p}'}(C_i) = n_{i,\mathfrak{p}'}$$

for each  $i \in \{1, 2, 3\}$  and every  $\mathfrak{p}' \notin S$ . Assume  $Q_i = [a_i : b_i]$  to be written in  $S$ -radical coprime coordinates for each  $i \in \{1, 2, 3\}$ . A point  $P = [x : y]$ , written in  $S$ -radical coprime coordinates too, belongs to the set in (4.2) if and only if there exist three units  $u_1, u_2, u_3 \in \sqrt{R_S^*}$  verifying the three following conditions

$$a_i y - b_i x = u_i C_i$$

for each  $i \in \{1, 2, 3\}$ . The three units  $u_1, u_2, u_3$  verify the following equation

$$(a_3 b_2 - b_3 a_2) C_1 u_1 + (b_3 a_1 - a_3 b_1) C_2 u_2 = (b_2 a_1 - b_1 a_2) C_3 u_3.$$

Now the proof is a trivial application of Beukers and Schlickewei's result (see Corollary 1) and algebraic manipulations.  $\square$

**Proposition 4.4** (Four-Point Lemma B). *Let  $K$  be a number field and  $S$  a finite set of places of  $K$  containing all the archimedean ones. Let  $Q_1, Q_2, Q_3, Q_4$  be four distinct points in  $\mathbb{P}_1(K)$ . The set*

$$\mathcal{F} = \{P \in \mathbb{P}_1(K) \mid \delta_{\mathfrak{p}}(P, Q_1) = \delta_{\mathfrak{p}}(P, Q_2), \delta_{\mathfrak{p}}(P, Q_3) = \delta_{\mathfrak{p}}(P, Q_4), \forall \mathfrak{p} \notin S\}$$

*is finite with cardinality bounded by  $C(3, |S|) + 2$ .*

*Proof.* We assume  $Q_i = [x_i : y_i]$  written in  $S$ -radical coprime coordinates for each  $i \in \{1, 2, 3\}$ . Without loss of generality we may assume  $Q_1 = [0 : 1]$  (up to a  $\sqrt{R_S}$ -invertible change of coordinates of  $\mathbb{P}_1$ ). Let  $P \in \mathcal{F}$  and we assume that  $P = [x : y]$  is written in  $S$ -radical coprime coordinates too. The condition  $\delta_{\mathfrak{p}}(P, Q_1) = \delta_{\mathfrak{p}}(P, Q_2)$  is equivalent to the existence of a unit  $u \in \sqrt{R_S^*}$  such that  $x = u(xy_2 - yx_2)$ , that is equivalent to

$$y = \frac{x}{ux_2}(uy_2 - 1).$$

Therefore  $P = [ux_2 : uy_2 - 1]$ . Since

$$\delta_{\mathfrak{p}}(P, Q_3) = v_{\mathfrak{p}}(ux_2y_3 - (uy_2 - 1)x_3) - \min\{v_{\mathfrak{p}}(ux_2), v_{\mathfrak{p}}(uy_2 - 1)\}$$

and

$$\delta_{\mathfrak{p}}(P, Q_4) = v_{\mathfrak{p}}(ux_2y_4 - (uy_2 - 1)x_4) - \min\{v_{\mathfrak{p}}(ux_2), v_{\mathfrak{p}}(uy_2 - 1)\}$$

are equal, there exists a unit  $v \in \sqrt{R_S^*}$  such that

$$ux_2y_3 - (uy_2 - 1)x_3 = v((ux_2)y_4 - (uy_2 - 1)x_4).$$

Therefore the two units  $u, v$  have to verify the following equation

$$(4.3) \quad Au + Bv + Cuv = 1$$

where  $A = \frac{x_2y_3 - x_3y_2}{-x_3}$ ,  $B = \frac{x_4}{x_3}$  and  $C = \frac{x_4y_2 - x_2y_4}{-x_3}$ .

We consider all possible vanishing subsums. We see that the cases  $Au = 0$ ,  $Bv = 0$ ,  $Cuv = 0$ ,  $Au + Cuv = 0$  and  $Bv + Cuv = 0$  are all impossible because the points  $Q_i$  are distinct and the case  $Au + Bv = 0$  provides two solutions for  $u$ . Hence  $u$  assume at most  $C(3, |S|) + 2$  possibilities and thus the cardinality of  $\mathcal{F}$  is bounded by  $C(3, |S|) + 2$ .  $\square$

**Proposition 4.5** (Three-Point Lemma C). *Let  $K$  be a number field and  $S$  a finite set of places of  $K$  containing all the archimedean ones. Let  $Q_1, Q_2, Q_3$  be three distinct points in  $\mathbb{P}_1(K)$ . The set*

$$\mathcal{T} = \{P \in \mathbb{P}_1(K) \mid \delta_{\mathfrak{p}}(P, Q_1) = \delta_{\mathfrak{p}}(P, Q_2) = \delta_{\mathfrak{p}}(P, Q_3), \mathfrak{p} \notin S\}$$

is finite with cardinality bounded by  $B(|S|)$ .

*Proof.* This is a particular case of the Three-Point Lemma B proven above.  $\square$

## 5. BOUNDS ON TAIL POINTS

The main objective of this section is to prove Theorem 1. To do so, we will prove four lemmas to bound the number of tail points of a  $K$ -rational periodic point. First we provide a bound for  $|\text{Tail}(\phi, K, P)|$  when  $P$  is a fixed  $K$ -rational point under the endomorphism  $\phi$ .

**Lemma 5.1.** *Let  $K$  be a number field and  $S$  a finite set of places of  $K$  containing all the archimedean ones. Let  $\phi$  be an endomorphism of  $\mathbb{P}^1$ , defined over  $K$ , and  $d \geq 2$  the degree of  $\phi$ . Assume  $\phi$  has good reduction outside  $S$ . Let  $P \in \mathbb{P}_1(K)$  be a fixed point of  $\phi$ . Then*

$$|\text{Tail}(\phi, K, P)| \leq L_1(d, |S|)$$

where  $L_1(d, |S|) = (d - 1)(1 + d(1 + B(|S|)))$ .

*Proof.* Consider  $P_1$  and  $P_2$  in  $\mathbb{P}_1(K)$  such that  $P_1$  is not periodic,  $\phi(P_1) = P$  and  $\phi(P_2) = P_1$ . If a point such as  $P_2$  do not exist then the cardinality of  $\text{Tail}(\phi, K, P)$  is bounded by  $d - 1$  and thus our bound holds.

Suppose  $\text{Tail}(\phi, K, P_2)$  is non empty and let  $Q \in \text{Tail}(\phi, K, P_2)$ . Lemma 2.9 implies that

$$\delta_{\mathfrak{p}}(Q, P_2) = \delta_{\mathfrak{p}}(Q, P_1) = \delta_{\mathfrak{p}}(Q, P) \quad \text{for every } \mathfrak{p} \notin S.$$

Therefore it is enough to apply Proposition 4.5 to get

$$|\text{Tail}(\phi, K, P_2)| \leq B(|S|).$$

Notice that if  $\text{Tail}(\phi, K, P_2)$  is empty then the previous inequality trivially holds.

Considering the fact that there are  $d - 1$  possibilities for  $P_1$  and  $d$  for  $P_2$ , we get

$$|\text{Tail}(\phi, K, P)| \leq (d - 1)(1 + d(1 + B(|S|)))$$

as desired.  $\square$

Now we provide a bound for  $|\text{Tail}(\phi, K, P)|$  when  $P$  is a  $K$ -rational periodic point of  $\phi$  of period 2.

**Lemma 5.2.** *Let  $K$  be a number field and  $S$  a finite set of places of  $K$  containing all the archimedean ones. Let  $\phi$  be an endomorphism of  $\mathbb{P}^1$ , defined over  $K$ , and  $d \geq 2$  the degree of  $\phi$ . Assume  $\phi$  has good reduction outside  $S$ . Let  $P \in \mathbb{P}_1(K)$  be a periodic point of  $\phi$  of period 2. Then*

$$|\text{Tail}(\phi, K, P)| \leq L_2(d, |S|)$$

where

$$L_2(d, |S|) = \max\{(2(C(3, |S|) + 2) + 1)d + 1, (d - 1)(1 + B(|S|)(B(|S|) + C(3, |S|) + 2 + 1))\}.$$

*Proof.* Let  $\mathfrak{p} \notin S$ . We denote  $P_1 = P$ , and  $P_2 = \phi(P)$ . If  $\text{Tail}(\phi, K, P)$  is empty then the result is trivially true. We assume that  $\text{Tail}(\phi, K, P)$  is not empty and we split the proof into two cases.

**Case 1: Suppose  $P_1$  has a unique non-periodic  $K$ -rational preimage.** Denote this preimage by  $Q \in \mathbb{P}_1(K)$ . Consider  $R \in \mathbb{P}_1(K)$  a preimage of  $Q$  and  $T \in \mathbb{P}_1(K)$  a tail point of  $R$ . If such point  $T$  does not exist then the cardinality of  $\text{Tail}(\phi, K, P)$  is bounded by  $d + 1$  and thus our bound holds.

We note that under application of  $\phi^2$ ,  $P_1$  and  $P_2$  are fixed, and  $R$  becomes a preimage of  $P_1$  and  $Q$  becomes a preimage of  $P_2$ ; the point  $T$  is (for  $\phi^2$ ) either a tail point of  $Q$  or a tail point of  $R$  depending on the parity of its tail length for  $\phi$ .

Without loss of generality (one can see that the situation is symmetric for  $\phi^2$ ), assume that  $T$  is a tail point for  $Q$  (under  $\phi^2$ ). By Lemma 2.9 we get  $\delta_{\mathfrak{p}}(T, Q) = \delta_{\mathfrak{p}}(T, P_2)$  and by Proposition 3.1 we get  $\delta_{\mathfrak{p}}(T, P_1) = \delta_{\mathfrak{p}}(Q, P_1) = 0$  and repeated application of Lemma 2.8 (on  $\phi^2$ ) we get  $\delta_{\mathfrak{p}}(T, R) \leq \delta_{\mathfrak{p}}(Q, P_1) = 0$ , which implies  $\delta_{\mathfrak{p}}(T, R) = 0$ .

Thus  $T$  satisfies the following two equations

$$\delta_{\mathfrak{p}}(T, Q) = \delta_{\mathfrak{p}}(T, P_2) \quad \delta_{\mathfrak{p}}(T, P_1) = \delta_{\mathfrak{p}}(T, R) \quad \text{for every } \mathfrak{p} \notin S.$$

By Lemma 4.4 there are at most  $C(3, |S|) + 2$  solutions for  $T$ . Then

$$|\text{Tail}(\phi^2, K, Q)| \leq C(3, |S|) + 2.$$

By symmetry we get

$$|\text{Tail}(\phi^2, K, R)| \leq C(3, |S|) + 2.$$

Considering that there are  $d$  possibilities for  $R$  we obtain

$$|\text{Tail}(\phi, K, P)| \leq (2(C(3, |S|) + 2) + 1)d + 1.$$

**Case 2: Suppose  $P_1$  has at least two non-periodic  $K$ -rational preimages.** There are at most  $d - 1$  such preimages. Denote two of them by  $Q_1$  and  $Q_2$  with the property that there is a point  $R \in \mathbb{P}_1(K)$  which is a preimage of  $Q_1$ , if such an  $R$  does not exist then  $|\text{Tail}(\phi, K, P)| \leq d - 1$ .

By 3.1 we get  $\delta_{\mathfrak{p}}(R, P_2) = \delta_{\mathfrak{p}}(Q_1, P_1) = 0$  and by Lemma 2.8 we get that  $\delta_{\mathfrak{p}}(R, Q_i) \leq \delta_{\mathfrak{p}}(Q_1, P_1) = 0$  for  $i \in \{1, 2\}$ . Thus  $R$  satisfies

$$\delta_{\mathfrak{p}}(R, Q_1) = \delta_{\mathfrak{p}}(R, Q_2) = \delta_{\mathfrak{p}}(R, P_2) = 0 \quad \text{for every } \mathfrak{p} \notin S.$$

By Lemma 4.5 we get there are at most  $B(|S|)$  solutions for  $R$ . Looking again at  $\phi^2$ , we see that  $Q_1$  and  $Q_2$  are preimages of the fixed point  $P_2$ . By similar arguments to the case 1,  $R$  has at most  $B(|S|) + C(3, |S|) + 2$  tail points (by summing up the cases of whether  $T$  is a tail point of  $R$  or  $Q_1$  under  $\phi^2$ ). Therefore we get

$$|\text{Tail}(\phi, K, P)| \leq (d-1)(1 + B(|S|)(B(|S|) + C(3, |S|) + 2 + 1))$$

Considering case 1 and case 2 we obtain

$$|\text{Tail}(\phi, K, P)| \leq \max\{(2(C(3, |S|) + 2) + 1)d + 1, (d-1)(1 + B(|S|)(B(|S|) + C(3, |S|) + 2 + 1))\}.$$

□

The next lemma provides a bound for  $|\text{Tail}(\phi, K, P)|$  when  $P$  is a  $K$ -rational periodic point of  $\phi$  of period 3.

**Lemma 5.3.** *Let  $K$  be a number field and  $S$  a finite set of places of  $K$  containing all the archimedean ones. Let  $\phi$  be an endomorphism of  $\mathbb{P}^1$ , defined over  $K$ , and  $d \geq 2$  the degree of  $\phi$ . Assume  $\phi$  has good reduction outside  $S$ . Let  $P \in \mathbb{P}_1(K)$  be a periodic point of  $\phi$  of period 3. Then*

$$|\text{Tail}(\phi, K, P)| \leq L_3(d, |S|),$$

where  $L_3(d, |S|) = ((1 + 3B(|S|))B(|S|) + 1)(d - 1)$ .

*Proof.* Let  $\mathfrak{p} \notin S$ . We denote  $P_1 = P$ ,  $P_2 = \phi(P)$ , and  $P_3 = \phi^2(P)$ . Consider  $Q, R \in \mathbb{P}_1(K)$  such that  $Q$  is not periodic,  $\phi(Q) = P_1$  and  $\phi(R) = Q$ . If such point  $R$  does not exist then the cardinality of  $\text{Tail}(\phi, K, P)$  is bounded by  $d - 1$  and thus our bound holds.

By 3.1 we get  $\delta_{\mathfrak{p}}(R, P_1) = \delta_{\mathfrak{p}}(R, P_3) = \delta_{\mathfrak{p}}(Q, P_1) = 0$  and by Lemma 2.8 we get that  $\delta_{\mathfrak{p}}(R, Q) \leq \delta_{\mathfrak{p}}(Q, P_1) = 0$ . Thus  $R$  satisfies

$$\delta_{\mathfrak{p}}(R, P_1) = \delta_{\mathfrak{p}}(R, P_3) = \delta_{\mathfrak{p}}(R, Q) = 0 \quad \text{for every } \mathfrak{p} \notin S.$$

By Lemma 4.5 we get there are at most  $B(|S|)$  solutions for  $R$ . Suppose  $\text{Tail}(\phi, K, R)$  is not empty. Let  $T \in \mathbb{P}_1(K)$  be a tail point of  $R$  and  $n$  its tail length for  $\phi$ .

We note that under application of  $\phi^3$ ,  $P_1, P_2$  and  $P_3$  are fixed, and  $R$  becomes a preimage of  $P_2$  and  $Q$  becomes a preimage of  $P_3$ ; the point  $T$  is (for  $\phi^3$ ) either a tail point of  $P_1$  or  $P_2$  or  $P_3$  depending if  $n$  is congruent to 1, 2 or 0 modulo 3, respectively.

**Case 1: Suppose  $n \equiv 0$  or  $2 \pmod{3}$ .**

One can see that the cases  $n \equiv 0 \pmod{3}$  and  $n \equiv 2 \pmod{3}$  are symmetric for  $\phi^3$ . We assume without loss of generality that  $n \equiv 0 \pmod{3}$  *i.e.*  $T$  is a tail point for  $Q$ . By Proposition 3.1 we get  $\delta_{\mathfrak{p}}(T, P_1) = \delta_{\mathfrak{p}}(T, P_2) = \delta_{\mathfrak{p}}(Q, P_2) = 0$  and repeated application of Lemma 2.8 (on  $\phi^3$ ) we get  $\delta_{\mathfrak{p}}(T, R) \leq \delta_{\mathfrak{p}}(Q, P_2) = 0$ , which implies  $\delta_{\mathfrak{p}}(T, R) = 0$ .

Thus  $T$  satisfies the following the equations

$$\delta_{\mathfrak{p}}(T, P_1) = \delta_{\mathfrak{p}}(T, P_2) = \delta_{\mathfrak{p}}(T, R) = 0 \quad \text{for every } \mathfrak{p} \notin S.$$

By Lemma 4.5 there are at most  $B(|S|)$  solutions for  $T$  for the case  $n \equiv 0 \pmod{3}$ , and thus  $2B(|S|)$  for both residue classes.

**Case 2: Suppose  $n \equiv 1 \pmod{3}$ .**

Since  $n \equiv 1 \pmod{3}$  we have that  $T$  is a tail point for  $P_1$ . By Proposition 3.1 we get  $\delta_{\mathfrak{p}}(T, P_2) = \delta_{\mathfrak{p}}(T, P_3) = \delta_{\mathfrak{p}}(\phi(T), P_1) = 0$  and applying Lemma 2.8 (on  $\phi$ ) we get  $\delta_{\mathfrak{p}}(T, Q) \leq \delta_{\mathfrak{p}}(\phi(T), P_1) = 0$ , which implies  $\delta_{\mathfrak{p}}(T, Q) = 0$ .

Thus  $T$  satisfies the following the equations

$$\delta_{\mathfrak{p}}(T, P_2) = \delta_{\mathfrak{p}}(T, P_3) = \delta_{\mathfrak{p}}(T, Q) = 0 \quad \text{for every } \mathfrak{p} \notin S.$$

By Lemma 4.5 there are at most  $B(|S|)$  solutions for  $T$ . Considering both cases we have that

$$|\text{Tail}(\phi, K, R)| \leq 3B(|S|).$$

Notice that if  $\text{Tail}(\phi, K, R)$  is empty then the previous inequality trivially holds. Considering that  $Q$  has at most  $d - 1$   $K$ -rational preimages and each of those preimages have at most  $B(|S|)$   $K$ -rational preimages we obtain

$$|\text{Tail}(\phi, K, P)| \leq ((1 + 3B(|S|))B(|S|) + 1)(d - 1).$$

□

The last lemma of this section provides a bound for  $|\text{Tail}(\phi, K, P)|$  when  $\phi$  admit a  $K$ -rational fixed point and a  $K$ -rational periodic point of period 2.

**Lemma 5.4.** *Let  $K$  be a number field and  $S$  a finite set of places of  $K$  containing all the archimedean ones. Let  $\phi$  be an endomorphism of  $\mathbb{P}^1$ , defined over  $K$ , and  $d \geq 2$  the degree of  $\phi$ . Assume  $\phi$  has good reduction outside  $S$ . Let  $P \in \mathbb{P}_1(K)$  be a fixed point of  $\phi$  and  $Q \in \mathbb{P}_1(K)$  a periodic point of  $\phi$  of period 2. Then*

$$|\text{Tail}(\phi, K, P)| \leq L_4(d, |S|)$$

where  $L_4(d, |S|) = (C(3, |S|) + 2 + 1)(d - 1)$ .

*Proof.* Let  $\mathfrak{p} \notin S$ . We denote  $Q_1 = Q$ , and  $Q_2 = \phi(Q)$ . Consider  $R \in \mathbb{P}_1(K)$  such that  $R$  is not periodic and  $\phi(R) = P$  and  $T \in \mathbb{P}_1(K)$  a tail point of  $R$ . If such point  $T$  does not exist then the cardinality of  $\text{Tail}(\phi, K, P)$  is bounded by  $d - 1$  and thus our bound holds.

By Lemma 2.9 we get  $\delta_{\mathfrak{p}}(T, R) = \delta_{\mathfrak{p}}(T, P_1)$  and by Proposition 3.1 we get  $\delta_{\mathfrak{p}}(T, Q_1) = \delta_{\mathfrak{p}}(T, Q_2) = 0$ . Thus  $T$  satisfies the following two equations

$$\delta_{\mathfrak{p}}(T, R) = \delta_{\mathfrak{p}}(T, P_1) \quad \delta_{\mathfrak{p}}(T, Q_1) = \delta_{\mathfrak{p}}(T, Q_2) \quad \text{for every } \mathfrak{p} \notin S.$$

By Lemma 4.4 there are at most  $C(3, |S|) + 2$  solutions for  $T$ . Considering that there are  $d - 1$  possibilities for  $R$  we obtain

$$|\text{Tail}(\phi, K, P)| \leq (C(3, |S|) + 2 + 1)(d - 1).$$

□

Before we prove Theorem 1 we emphasize that  $L_1(d, |S|)$  is quadratic in terms of  $d$  but  $L_2(d, |S|)$ ,  $L_3(d, |S|)$ , and  $L_4(d, |S|)$  are linear in terms of  $d$ . We will denote by  $CV(d, |S|)$  the bound for  $|\text{Per}(\phi, K)|$  mentioned in [CV, Corollary 1] which is linear in terms of  $d$  and we denote by  $T(|S|)$  the refined result of Troncoso's bound proven in Theorem 2 part (2).

Now we have all the tools to prove the main theorem of this paper.

*Proof of Theorem 1.* If the map  $\phi$  has more than three periodic points in  $\mathbb{P}_1(K)$ , then we have

$$|\text{PrePer}(\phi, K)| \leq T(|S|) + CV(d, |S|).$$

If the map  $\phi$  has at most three periodic points in  $\mathbb{P}_1(K)$ , we have two cases. First if  $\phi$  has a  $K$ -rational point of period at least 2 then

$$|\text{PrePer}(\phi, K)| \leq \max\{L_4(d, |S|) + 2L_2(d, |S|), 3L_3(d, |S|)\} + 3.$$

On the other hand, if  $\phi$  has no  $K$ -rational points of period at least 2 then

$$|\text{PrePer}(\phi, K)| \leq 3L_1(d, |S|) + 3.$$

□

## 6. APPENDIX

For the convenience of the reader in this section we give all the bounds used in this paper in an explicit form. Let  $S$  be a finite set of places of a number field  $K$  containing all the archimedean ones. Let  $n$  be a non negative integer and  $d$  a positive integer.

$$\begin{aligned} B(|S|) &= 2^{16|S|}; \\ C(n, |S|) &= e^{(6n)^{3n}(n|S|+1-n)}; \\ L_1(d, |S|) &= (d-1)(1+d(1+B(|S|))); \\ L_2(d, |S|) &= \max\{(2(C(3, |S|)+2)+1)d+1, (d-1)(1+B(|S|)(B(|S|)+C(3, |S|)+2+1))\}; \\ L_3(d, |S|) &= ((1+3B(|S|))B(|S|)+1)(d-1); \\ L_4(d, |S|) &= (C(3, |S|)+2+1)(d-1); \\ CV(d, |S|) &= (3B(|S|)+13)d+27B(|S|)+C(5, |S|)+6C(3, |S|)+32; \\ T(|S|) &= 12 \cdot 7^{4|S|}; \\ L(d, |S|) &= \max\{T(|S|)+CV(d, |S|), L_4(d, |S|)+2L_2(d, |S|)+3, 3L_3(d, |S|)+3\}; \\ Q(d, |S|) &= \max\{T(|S|)+CV(d, |S|), 3L_1(d, |S|)\}. \end{aligned}$$

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