# Research Statement

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My research interests lie primarily in arithmetic dynamics with much of my work motivated by Morton-Silverman's Uniform Boundedness Conjecture (UBC). Arithmetic dynamics lies at the intersection of dynamical systems and number theory. Primary topics include the study of the iteration of polynomial or rational functions on integer, rational, $p$-adic, and algebraic points.

I have organized this research statement into four sections. In the introduction I give basic notation and motivation of arithmetic dynamics. I also quote the latest results in the area which are relevant to my research. In the second section, I provide details about my previous work which has inspired new projects. In the third section, I explain the three projects on which I am currently working on, which are each at different stages of completion. Finally in the last section, I provide a few ideas for research projects for undergraduates students. Each of these is appropriate for upper class students, but they may be modified to work for freshmen or sophomores.

# 1 Introduction

Let $K$ be a number field and $\mathbb{P}^N$ the $N$-dimensional projective space. Let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be an endomorphism of degree $d \geq 2$ defined over $K$. The *orbit* of $P \in \mathbb{P}^N(K)$ under $\phi$ is the set $O_\phi(P) = \{\phi^n(P) : n \geq 0\}$. A point $P \in \mathbb{P}^N(K)$ is called *periodic* under $\phi$ if there is an integer $n > 0$ such that $\phi^n(P) = P$; the minimal such $n$ is called the *period* of $P$. It is called *preperiodic* under $\phi$ if there is an integer $m \geq 0$ such that $\phi^m(P)$ is periodic. A point that is preperiodic but not periodic is called a *tail* point. Let $\mathrm{Tail}(\phi, K)$, $\mathrm{Per}(\phi, K)$ and $\mathrm{PrePer}(\phi, K)$ be the sets of $K$-rational tail, periodic and preperiodic points of $\phi$, respectively.

There are many challenging questions in arithmetic dynamics. A natural and central question in the subject is

**Question 1.** *How many $K$-rational preperiodic points does $\phi$ have? That is, how large is $|\mathrm{PrePer}(\phi, K)|$? If $|\mathrm{PrePer}(\phi, K)|$ is finite can one give a bound for it?*

Of course Question 1 is not interesting in the complex setting, since there are always infinitely many complex preperiodic points. However in the number field case, Northcott [12] proved in 1950 that the total number of $K$-rational preperiodic points of $\phi$ is finite. In fact, Northcott's theorem holds for an endomorphism of $\mathbb{P}^N$, defined over $K$ and from Northcott's proof, an explicit bound can be found in terms of the coefficient of $\phi$. The problem is to find a bound on the number of preperiodic points that depends in a "minimal" way on the map $\phi$. In 1994, Morton and Silverman [11] conjectured the celebrated Uniform Boundedness Conjecture that we state below.

**Conjecture 1.1** (Uniform Boundedness Conjecture (UBC) ).
*Let $K$ be a number field with $[K : \mathbb{Q}] = D$, and let $\phi$ be an endomorphism of $\mathbb{P}^N$, defined over $K$. Let $d \geq 2$ be the degree of $\phi$. Then there is $C = C(D, N, d)$ such that $\phi$ has at most $C$ preperiodic points in $\mathbb{P}^N(K)$.*

This conjecture is an extremely strong uniformity conjecture. For example, the UBC for maps of degree 4 on $\mathbb{P}^1$ defined over $\mathbb{Q}$ implies Mazur's theorem [9] that the torsion subgroup of an elliptic curve $E/\mathbb{Q}$ is bounded independently of $E$. More generally, the UBC for maps of degree 4 on $\mathbb{P}^1$ defined over a number field $K$ implies Merel's theorem [10] that the size of the torsion subgroup of an elliptic curve over a number field $K$ is bounded only in terms of the degree of $[K : \mathbb{Q}]$. The conjecture can also be similarly applied to uniform boundedness of torsion subgroups of Lattès maps and abelian varieties; for more detail see [7].

The Uniform Boundedness Conjecture seems extremely difficult to prove even in the simplest case when $(K, N, d) = (\mathbb{Q}, 1, 2)$. A natural relaxation of the uniform boundedness conjecture is the consideration of the set of primes for which a rational map $\phi$ has bad reduction. A rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$ defined over a number field $K$ is said to have *good reduction* at a prime $\mathfrak{p}$ of $K$ if $\phi$ can be written as $\phi = [F(X,Y) : G(X,Y)]$ where $F, G \in R_{\mathfrak{p}}[X,Y]$ are homogeneous polynomials of degree $d$, such that the resultant of $F$ and $G$ is a $\mathfrak{p}$-unit, where $R_{\mathfrak{p}}$ is the localization of the ring of integers of $K$ at $\mathfrak{p}$. The map $\phi$ is said to have *bad reduction* at a prime $\mathfrak{p}$ of $K$ if $\phi$ does not have good reduction at $\mathfrak{p}$. For a fixed finite set $S$ of places of $K$ containing all the Archimedean ones, we say that $\phi$ has good reduction outside of $S$ if it has good reduction at each place $\mathfrak{p} \notin S$.

In the special case of rational functions $\phi : \mathbb{P}^1 \to \mathbb{P}^1$, there are several results giving a uniform bound on the number of periodic/preperiodic points of $\phi$ depending on the cardinality of a finite set of places $S$, which includes all Archimedean places, together with the constants $[K : \mathbb{Q}]$ and $\deg(\phi)$, under the assumption that $\phi$ has good reduction outside of $S$. Below we briefly cite the latest effective bound for the set of $K$-rational preperiodic points.

If $\phi$ is a polynomial map of degree $d \geq 2$ then Benedetto [1] proved in 2007 that $|\operatorname{PrePer}(\phi, K)|$ is bounded by $O(|S| \log |S|)$, where big-$O$ is essentially $\frac{d^2-2d+2}{\log d}$ for large $|S|$. This is the best bound known for the polynomial case.

In the general case, if $\phi$ is a rational function defined over a number field $K$, Canci and Paladino [5] proved in 2014 that the length of finite orbits is bounded by

$$\max\left\{(2^{16|S|-8}+3)[12|S|\log(5|S|)]^{[K:\mathbb{Q}]}, [12(|S|+2)\log(5|S|+5)]^{4[K:\mathbb{Q}]}\right\}. \tag{1}$$

The bounds mentioned in (1) can be used to deduce a bound on $|\operatorname{PrePer}(\phi, K)|$. Indeed, the bound deduced from (1) is roughly of the order of $d^{2^{16|S|}(|S|\log(|S|)^{[K:\mathbb{Q}]}}$, where $d \geq 2$ is the degree of $\phi$. This bound is polynomial in the degree of $\phi$, however it will be rather large in terms of $|S|$. It is important to mention that in the work of Canci and Paladino [5] it is not only true for number fields but also for global fields.

In 2016, Canci and Vishkautsan [6] proved a bound for $|\operatorname{Per}(\phi, K)|$ of order $d \cdot 2^{16|S|} + 2^{3^7|S|}$. Their bound is outstanding in terms of $d$; however, they did not provide a bound for the cardinality of the set of $K$-rational preperiodic points.

In 2017, I [14] was able to obtain $|\operatorname{PrePer}(\phi, K)| \leq 5(2^{16|S|d^3}) + 3$ which is a significant improvement of previous known bounds on the cardinality of the set of $K$-rational preperiodic points of $\phi$.

# 2 Previous Work

My doctoral studies were completed under the direction of Aaron Levin at Michigan State University. With him I started working on problems in arithmetic dynamics, in particular on some papers from Canci [3] and [4]. During that time, I published a paper in the *Journal of Number*

*Theory* [14]. Most of the techniques introduced in that paper have led to new research projects. In this section we state the main tools and theorem proven in [14].

We recall the definition of the logarithmic $\mathfrak{p}$-adic chordal distance on $\mathbb{P}^1(K)$ for a finite place $\mathfrak{p}$ of a number field $K$. Let $P_1 = [x_1 : y_1]$ and $P_2 = [x_2 : y_2]$ be points in $\mathbb{P}^1(K)$. We denote by

$$\delta_{\mathfrak{p}}(P_1, P_2) = v_{\mathfrak{p}}(x_1y_2 - x_2y_1) - \min\{v_{\mathfrak{p}}(x_1), v_{\mathfrak{p}}(y_1)\} - \min\{v_{\mathfrak{p}}(x_2), v_{\mathfrak{p}}(y_2)\}$$

the $\mathfrak{p}$-*adic logarithmic chordal distance* between the points $P_1$ and $P_2$.

I was able to identify and prove an important arithmetic relation between $K$-rational tail points and $K$-rational periodic points.

**Proposition 2.1.** *Let $\phi$ be an endomorphism of $\mathbb{P}^1$, defined over $K$. Suppose $\phi$ has good reduction outside $S$. Let $R \in \mathbb{P}^1(K)$ be a tail point and let $n$ be the period of the periodic part of the orbit of $R$. Let $P \in \mathbb{P}^1(K)$ be any periodic point that is not $\phi^{mn}(R)$ for some $m$. Then* $\forall \mathfrak{p} \notin S \quad \delta_{\mathfrak{p}}(P, R) = 0$. i.e. *$P$ is $S$-integral with respect to $R$.*

The previous proposition has strong consequences. For instance, under some mild hypotheses we get bounds for $|\operatorname{Tail}(\phi, K)|$ and $|\operatorname{Per}(\phi, K)|$ *independent* of the degree of the rational function.

**Theorem 2.2.** *Let $K$ be a number field and $S$ a finite set of places of $K$ containing all the Archimedean ones. Let $\phi$ be an endomorphism of $\mathbb{P}^1$, defined over $K$ and $d \geq 2$ the degree of $\phi$. Assume $\phi$ has good reduction outside $S$.*

*(a) If there are at least three $K$-rational tail points of $\phi$ then*

$$|\operatorname{Per}(\phi, K)| \leq 2^{16|S|} + 3.$$

*(b) If there are at least four $K$-rational periodic points of $\phi$ then*

$$|\operatorname{Tail}(\phi, K)| \leq 4(2^{16|S|}).$$

A direct corollary of the previous theorem gives $|\operatorname{PrePer}(\phi, K)| \leq 5(2^{16|S|d^3}) + 3$.

# 3 Current Projects

In this section I describe three on-going projects. Each of them is at a different stage of progress.

**Project 1. Scarcity of finite orbits for rational functions over a number field.** *Joint work with J. Canci and S. Vishkautsan. The paper of Canci and Vishkautsan [6] and my paper [14] have similar ideas, and they naturally complement each other. For this reason we decided to start a project together. We are currently in the final stage of this project. We are making the last reviews before submitting our paper to a journal. The main theorem we prove is the following:*

**Theorem 3.1.** *Let $K$ be a number field and $S$ a finite set of places of $K$ containing all the Archimedean ones. Let $\phi$ be an endomorphism of $\mathbb{P}^1$, defined over $K$, and $d \geq 2$ the degree of $\phi$. Assume $\phi$ has good reduction outside $S$. Then*

$$|\operatorname{PrePer}(\phi, K)| \leq Q(|S|, d) = \alpha_1 d^2 + \beta_1 d + \gamma_1,$$

*where $\alpha_1$, $\beta_1$ and $\gamma_1$ are positive integers depending only on $|S|$ and can be effectively computed.*

*In addition, if we assume that $\phi$ has a $K$-rational periodic point of period at least two then*

$$|\operatorname{PrePer}(\phi, K)| \leq L(|S|, d) = \alpha_2 d + \beta_2,$$

*where $\alpha_2$ and $\beta_2$ are positive integers depending only on $|S|$ and can be effectively computed.*

*Compared with previous results the bound obtained in the previous theorem is outstanding, just to give an idea to the reader we can roughly estimate $Q(|S|, d)$ and $L(|S|, d)$ as $2^{3^{10|S|}} \cdot d^2$ and $2^{3^{10|S|}} \cdot d$, respectively.*

*In the paper, we also provide a refinement of Theorem 2.2 and two extremely useful lemmas called the three-point and four-point lemma.*

**Project 2. Preperiodic hypersurfaces and preperiodic points.** *The main idea of this work is to generalize my results from [14] in higher dimensions. In particular my goal is to get an effective results for the cardinality of the set of $K$-preperiodic hypersurfaces of $\mathbb{P}^n$. This project is currently at the middle stage of progress. I already have strong tools and some partial results.*

*I define the concepts of $K$-rational periodic, preperiodic, and tail hypersurfaces, and I generalized the logarithmic $\mathfrak{p}$-adic chordal distance between two points in $\mathbb{P}^1$ to a logarithmic $\mathfrak{p}$-adic distance between a $K$-rational hypersurface and a point in $\mathbb{P}^n$.*

*This new distance is denoted by $\delta_{\mathfrak{p}}(P; H)$ where $P \in \mathbb{P}^n(K)$ and $H$ is a hypersurface of $\mathbb{P}^n$ defined over $K$. Using the previous terms, I was able to generalize Proposition 2.1 to obtain an arithmetic relation between $K$-rational tail points and $K$-rational periodic points to a relation between $K$-rational tail hypersurfaces and $K$-rational periodic points.*

**Theorem 3.2.** *Let $\phi$ be an endomorphism of $\mathbb{P}^n$, defined over $K$. Suppose $\phi$ has good reduction outside $S$. Let $H$ be a $K$-rational tail hypersurface, $m$ the period of the periodic part of the orbit of $H$, and $H'$ the periodic hypersurface such that $H' = \phi^{m_0 m}(H)$ for some $m_0 > 0$. Let $P \in \mathbb{P}^n(K)$ be any periodic point such that $P \notin supp\{H'\}$. Then $\delta_v(P; H) = 0$ for every $v \notin S$.*

*Let $\phi$ be an endomorphism of $\mathbb{P}^n$ over $K$, $S$ be the set of places of bad reduction for $\phi$ and $HTail(\phi, K, e)$ be the set of $K$-rational tail hypersurfaces of $\mathbb{P}^n$ of degree $e$. If we consider $N = \binom{e+n}{e} - 1$ and assume that $\phi$ has at least $2N+1$ $K$-rational periodic points such that none $N+1$ of them lie in a hypersurface of degree $e$ then we give an effective bound on a large subset of $HTail(\phi, K, e)$ depending on $e$ and the number of places of bad reduction $|S|$. Finally, we prove that the set $HTail(\phi, K, e)$ is finite if we assume that $\phi$ is an endomorphism of $\mathbb{P}^2$.*

**Project 3. Bounds for preperiodic points for rational maps with good reduction over a global field.** *Joint work with H. Moreno. The main idea of this work is to generalize my results from [14] to a field $K$ with positive characteristic. This project is at the earliest stage of progress. We have clear ideas of what we would like to prove and how we will approach those ideas.*

*Concretely, in this project we want to generalize Proposition 2.1 to obtain an arithmetic relation between $K$-rational tail points and $K$-rational periodic points. After that we are confident we could use some techniques from Canci and Paladino's paper to obtain a bound for the set of $K$-rational preperiodic points. Furthermore, if we obtain a result like the 3-point lemma or 4-point lemma for $K$ then we may be able to provide a bound for the set of $K$-rational preperiodic points which will be polynomial on the degree of the endomorphism.*

# 4 Courses and Projects for Undergraduate

I have always found it exciting to share my knowledge in mathematics with others. I would like to participate in math clubs and math discussions and to give talks and seminars to promote mathematics and arithmetic dynamics among students and faculty.

This winter term (2017/2018) Dr. Mullins and I will be supervising undergraduate research projects for senior students in Mathematics. I will work with three groups of students, conducting research on knot invariants, on an upper bound for the rope length on pretzel knots and on information theory with an application in genomics.

In the future I would like to continue working with undergraduate students on research projects. In fact, my research can be used as a base for several interesting projects. Therefore I would like to introduce undergraduate students to my area of expertise, arithmetic dynamics. There are two ways I would like do so. The first way is to teach a course in arithmetic dynamics. In this course we would cover the basics of projective spaces, the big ideas of dynamical systems, and how these ideas can be approached with geometry and number theory tools. The only prerequisite for such a course would be introduction to proofs. For this reasons, this course would be aimed at math majors or upper class students.

The second way I would introduce students to my research would be with undergraduate research projects. Below I give three examples of projects that I envision for undergraduate students. These three example are proposed for a senior student, but they can be easily modified to work for students with all kinds of backgrounds, even first-years.

1. Height functions are a fundamental tool in arithmetic dynamics and arithmetic geometry. A good project for an undergraduate student would be to understand the definition of height functions over a number field and compute the height of $\sqrt{p}$ in the the number field $K = \mathbb{Q}(\sqrt{p})$. For more advanced students, I would also ask them to understand the height function over a divisor.

2. Let $K$ be a number field and $S$ a finite set of places of $K$ containing all the Archimedean ones. We consider the $S$-unit equation $ax + by = 1$ where $a, b \in K^*$ and $x, y$ are $S$-units. Bounds on the number of solutions of these equations yield powerful consequences in different areas of mathematics. A great undergraduate project would be to learn how to use the $S$-unit equation and to understand the different applications that the $S$-unit equation has inside and outside number theory.

3. Arithmetic dynamics can be studied using techniques in number theory as well as using techniques in computer science. For students interested in mathematics and computer science, I would suggest the following project: Program an algorithm in Sage or Magma to find the preperiodic points of a quadratic polynomial $f_c(x) = x^2 + c$. Use your algorithm or the ones known to study the graph of these quadratic polynomials. Could you conjecture a bound for the set of preperiodic points?.

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