K-RATIONAL PREPERIODIC POINTS AND HYPERSURFACES ON PROJECTIVE SPACE.

By

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics — Doctor of Philosophy

2017

ABSTRACT

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The present thesis has two main parts. In the first one, we study bounds for the number of rational preperiodic points of an endomorphism of $\mathbb{P}^1$. Let K be a number field and ϕ be an endomorphism of $\mathbb{P}^1$ over K of degree $d \geq 2$. Let S be the set of places of bad reduction for ϕ (including the archimedean places). Let $\mathrm{Per}(\phi, K)$, $\mathrm{PrePer}(\phi, K)$, and $\mathrm{Tail}(\phi, K)$ be the set of K-rational periodic, preperiodic, and purely preperiodic points of ϕ, respectively.

If we assume that $|\mathrm{Per}(\phi, K)| \geq 4$ (resp. $|\mathrm{Tail}(\phi, K)| \geq 3$), we prove bounds for $|\mathrm{Tail}(\phi, K)|$ (resp. $|\mathrm{Per}(\phi, K)|$) that depend only on the number of places of bad reduction $|S|$ (and not on the degree d). We show that the hypotheses of this result are sharp, giving counterexamples to any possible result of this form when $|\mathrm{Per}(\phi, K)| < 4$ (resp. $|\mathrm{Tail}(\phi, K)| < 3$). The key tool involved in these results is a bound for the number of solutions of S-unit equations.

Using bounds for the number of solutions of the celebrated Thue-Mahler equation, we obtain bounds for $|\mathrm{Per}(\phi, K)|$ and $|\mathrm{Tail}(\phi, K)|$ in terms of the number of places of bad reduction $|S|$ and the degree d of the rational function ϕ. Bounds obtained in this way are a significant improvement to previous result given by J. Canci and L. Paladino.

In the second part of the thesis, we study the set of K-rational purely preperiodic hypersurfaces of $\mathbb{P}^n$ of a given degree for an endomorphism of $\mathbb{P}^n$. Let ϕ be an endomorphism of $\mathbb{P}^n$ over K, S be the set of places of bad reduction for ϕ and $\mathrm{HTail}(\phi, K, e)$ be the set of

K-rational purely preperiodic hypersurfaces of $\mathbb{P}^n$ of degree e.

We give a strong arithmetic relation between K-rational purely preperiodic hypersurfaces and K-rational periodic points. If we consider $N = \binom{e+n}{e} - 1$ and assume that ϕ has at least $2N+1$ K-rational periodic points such that no $N+1$ of them lie in a hypersurface of degree e then we give an effective bound on a large subset of $\mathrm{HTail}(\phi, K, e)$ depending on e and the number of places of bad reduction $|S|$. Finally, we prove that the set $\mathrm{HTail}(\phi, K, e)$ is finite if we assume that ϕ is an endomorphism of $\mathbb{P}^2$.

To my daughter Emma, my son Sebastian and my wife Yira.
I love you all.
I would not be here without you.

ACKNOWLEDGMENTS

I have received a tremendous amount of support during my time at Michigan State University. I am truly grateful for my advisor Dr. Aaron Levin, there are not words to describe how thankful I am to be your student.

During my studies, I was able to dedicate more time to research because I was partially funded by my advisor grants. Additionally, I received a great deal of support from Michigan State University, which has helped me in completing this project: College of Natural Science, Department of Mathematics, Becas Chile and Dissertation Completion Fellowship.

Many professors help me to obtain my doctorate degree. In particular, I want to thank Herminia Ochsenius, Martin Berz, Rajesh Kulkarni, and Aaron Levin.

My friends, both in and out of the math department, played a key role throughout this process. I especially want to thank Casey Machen, Charlotte Ure, Hector Moreno, and Robert Auffarth.

My family is the core of my life. There were always present to support me, love me and keep me sane. I want to thanks my mom Emma Naranjo, my dad Marco Troncoso, my sisters Natalia Troncoso and Paula Troncoso, my wife Yira Feliciano, and my children Sebastian Troncoso and Emma Troncoso.

Those not mentioned here are not forgotten. I am thankful for everyone else who has helped me along the way.

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KEY TO SYMBOLS

1. $\mathbb{N}$ the set of natural numbers.
2. $\mathbb{N}_0$ the set of non-negative integers.
3. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ the set of integers, rational, real and complex numbers respectively.
4. $\subset$ means subset.
5. $\subsetneq$ means a proper subset.
6. $|A|$ the cardinality of a set A.
7. $\mathbf{i} = (i_0, \ldots, i_n) \in \mathbb{N}_0^{n+1}$ an $n+1$-dimensional multi-index.
8. $|\mathbf{i}| = i_0 + \cdots + i_n$
9. $\mathbf{X} = (X_0, \ldots, X_n)$ where $X_0, \ldots, X_n$ are $n+1$ variables.
10. $\mathbf{X}^{\mathbf{i}} = X_0^{i_0} \cdots X_n^{i_n}$
11. R^* the group of units of a ring R.
12. K a number field.
13. $\bar{K}$ an algebraic closure of K.
14. $\mathcal{O}$ the ring of integers of K.
15. $\mathfrak{p}$ a non-zero prime ideal of $\mathcal{O}$.
16. $v_{\mathfrak{p}}$ the $\mathfrak{p}$-adic valuation on K corresponding to the prime ideal $\mathfrak{p}$ (we always assume $v_{\mathfrak{p}}$ to be normalized so that $v_{\mathfrak{p}}(K^*) = \mathbb{Z}$).
17. If the context is clear, we will also use $v_{\mathfrak{p}}(I)$ for the $\mathfrak{p}$-adic valuation of a fractional ideal I of K.
18. S a fixed finite set of places of K including all archimedean places.
19. $|S| = s$ the cardinality of S.
20. $\mathcal{O}_S = \{x \in K : v_{\mathfrak{p}}(x) \geq 0$ for every prime ideal $\mathfrak{p} \notin S\}$ the ring of S-integers.
21. $\mathcal{O}_S^* = \{x \in K : v_{\mathfrak{p}}(x) = 0$ for every prime ideal $\mathfrak{p} \notin S\}$ the group of S-units.

- Let ϕ be an endomorphism of $\mathbb{P}^n$ defined over K.

22. $\mathrm{Per}(\phi, K)$ the set of K-rational periodic points.

23. $\mathrm{Tail}(\phi, K)$ the set of K-rational tail points.

24. $\mathrm{PrePer}(\phi, K)$ the set of K-rational preperiodic points.

25. $\mathrm{HPer}(\phi, K, e)$ the set of K-rational periodic hypersurfaces of degree e.

26. $\mathrm{HTail}(\phi, K, e)$ the set of K-rational tail hypersurfaces of degree e.

27. $\mathrm{HPrePer}(\phi, K, e)$ the set of K-rational preperiodic hypersurfaces of degree e.

28. $\mathrm{HPer}(\phi, K)$ the set of K-rational periodic hypersurfaces.

29. $\mathrm{HTail}(\phi, K)$ the set of K-rational tail hypersurfaces.

30. $\mathrm{HPrePer}(\phi, K)$ the set of K-rational preperiodic hypersurfaces.

Chapter 1

Introduction

Let $\mathcal{S}$ be a set and $\phi\colon \mathcal{S} \to \mathcal{S}$ a function mapping the set $\mathcal{S}$ to itself. A (discrete) dynamical system is a pair consisting of the set $\mathcal{S}$ and the function ϕ. We denote by ϕ^n the nth iterate of ϕ under composition and by ϕ^0 the identity map. The *orbit* of $P \in \mathcal{S}$ under ϕ is the set $O_\phi(P) = \{\phi^n(P) : n \geq 0\}$.

The set $\mathcal{S}$ could be simply a set with no additional structure but most frequently we study dynamics when the set $\mathcal{S}$ has some additional structure. In arithmetic dynamics we are interested when the set $\mathcal{S}$ is an arithmetic set such as $\mathbb{Z}$, $\mathbb{Q}$, number fields K, quasi-projective variety, K-rational points, etc. and the function ϕ is a polynomial, a rational map, an endomorphism, etc. In this arithmetic context the **Principal Goal of Dynamics** is to classify the points P in $\mathcal{S}$ according to the behavior of their orbits $O_\phi(P)$ when $\mathcal{S}$ is an arithmetic set.

Let K be a number field. Our study in arithmetic dynamics will be when $\mathcal{S}$ is $\mathbb{P}^N(K)$ and ϕ an endomorphism of $\mathbb{P}^N$ of degree $d \geq 2$. A point $P \in \mathbb{P}^N(K)$ is called *periodic* under ϕ if there is an integer $n > 0$ such that $\phi^n(P) = P$. It is called *preperiodic* under ϕ if there is an integer $m \geq 0$ such that $\phi^m(P)$ is periodic. A point that is preperiodic but not periodic is called a *tail* point. Let $\mathrm{Tail}(\phi, K)$, $\mathrm{Per}(\phi, K)$ and $\mathrm{PrePer}(\phi, K)$ be the sets of K-rational tail, periodic and preperiodic points of ϕ, respectively.

A first objective of this thesis is to study the cardinality of the sets $\mathrm{Tail}(\phi, K)$, $\mathrm{Per}(\phi, K)$

and $\mathrm{PrePer}(\phi, K)$. We start by asking

- Is the set of K-rational preperiodic points finite or infinite?.
- If finite, can we give an effective bound?.

Northcott [Nor50] proved in 1950 that the total number of K-rational preperiodic points of ϕ is finite. In fact, from Northcott's proof, an explicit bound can be found in terms of the coefficients of ϕ, the number field K and the dimension N.

Even when Northcott answered both questions, a bound for $\mathrm{PrePer}(\phi, K)$ in terms of only a few basic parameters is desired. In 1994, Morton and Silverman [MS94] conjectured the celebrated Uniform Boundedness Conjecture (UBC) which predicts the existence of such a bound depending only on d, the dimension of the projective space and the degree of K.

Conjecture 1.0.1 (Uniform Boundedness Conjecture)**.**
Let K be a number field with $[K : \mathbb{Q}] = D$, and let ϕ be an endomorphism of $\mathbb{P}^N$, defined over K. Let $d \geq 2$ be the degree of ϕ. Then there is $C = C(D, N, d)$ such that ϕ has at most C preperiodic points in $\mathbb{P}^N(K)$.

This conjecture is an extremely strong uniformity conjecture. For example, the UBC on maps of degree 4 on $\mathbb{P}^1$ defined over $\mathbb{Q}$ implies Mazur's theorem that the torsion subgroup of an elliptic curve $E/\mathbb{Q}$ is bounded independently of E. More generally, the UBC for maps of degree 4 on $\mathbb{P}^1$ defined over K implies Merel's theorem that the size of the torsion subgroup of an elliptic curve over a number field K is bounded only in terms of the degree of $[K : \mathbb{Q}]$. The conjecture can also be applied to Lattès maps and abelian varieties, for more detail see [Fak01], [Maz77] and [Mer96].

Poonen [Poo98] later stated a sharper version of the conjecture for the special case of quadratic polynomials over $\mathbb{Q}$. Since every such quadratic polynomial map is conjugate to a polynomial of the form $\psi_c(z) = z^2 + c$ with $c \in \mathbb{Q}$ we can state Poonen's conjecture as follows:

Conjecture 1.0.2 (Poonen's conjecture)**.**
Let $\psi_c \in \mathbb{Q}[z]$ be a polynomial of degree 2 of the form $\psi_c(z) = z^2 + c$ with $c \in \mathbb{Q}$. Then

$$|\operatorname{PrePer}(\psi_c, \mathbb{Q})| \leq 9.$$

Even though Poonen's Conjecture is arguably the simplest case of the UBC, a proof of Poonen's Conjecture seems to be very far off at this time. If we consider polynomials of the form $\psi_c(z) = z^2 + c$ with $c \in \mathbb{Q}$, B. Hutz and P. Ingram [HI13] have shown that Poonen's conjecture holds when the numerator and denominator of c don't exceed 10^8. For more information on quadratic rational functions see [BCH$^+$14], [Can10], [FHI$^+$09], [Man07], [MN06], [Poo98].

Even though the UBC or Poonen's conjecture are impossible to prove at the moment, if we allow the bound from the UBC to depend on one more parameter, then effective results can be given in the case of $\mathbb{P}^1$.

In the first half of the thesis we work in the case $N = 1$, so from now we assume that ϕ is an endomorphism of $\mathbb{P}^1$. Let S be the set of places of K at which ϕ has bad reduction, including all archimedean places of K. The nonarchimedean places of bad reduction are those in which the degree of the reduction of ϕ in the residue field decreases. In other words, a place is said to be a place of good reduction if ϕ has a good behavior in the residue field associated with the place. Then the extra parameter needed to give effective results is the cardinality of S.

The first main result of this thesis [[Tro], Corollary 1.3.] gives a bound for $|\operatorname{PrePer}(\phi, K)|$

in terms of the number of places of bad reduction $|S|$ and the degree of the rational function ϕ. This bound significantly improves a previous bound given by J. Canci and L. Paladino [CP16].

In the second result, assuming that $|\operatorname{Tail}(\phi, K)| \geq 3$ (resp. $|\operatorname{Per}(\phi, K)| \geq 4$), we prove bounds for $|\operatorname{Per}(\phi, K)|$ (resp. $|\operatorname{Tail}(\phi, K)|$) that depend only on the number of places of bad reduction $|S|$ and $[K : \mathbb{Q}]$ (and not on the degree of ϕ). We show that the hypotheses of this result are sharp. **??** and **??** give counterexamples to any possible result of this form when $|\operatorname{Tail}(\phi, K)| < 3$ (resp. $|\operatorname{Per}(\phi, K)| < 4$).

Theorem 1.0.3. *Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S.*

(a) If there are at least three K-rational tail points of ϕ then

$$|\operatorname{Per}(\phi, K)| \leq 2^{16|S|} + 3.$$

(b) If there are at least four K-rational periodic points of ϕ then

$$|\operatorname{Tail}(\phi, K)| \leq 4(2^{16|S|}).$$

Using the previous theorem, we can deduce a bound for $|\operatorname{PrePer}(\phi, K)|$ in terms of $|S|$ and the degree of ϕ for any endomorphism of $\mathbb{P}^1$.

Corollary 1.0.4. *Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S. Then* *(a)* $|\operatorname{Per}(\phi, K)| \leq 2^{16|S|d^3} + 3$.

(b) $|\operatorname{Tail}(\phi, K)| \leq 4(2^{16|S|d^3})$.

(c) $|\operatorname{PrePer}(\phi, K)| \leq 5(2^{16|S|d^3}) + 3$.

These bounds depend, ultimately, on a reduction to S-unit equations. Using a reduction to Thue-Mahler equations instead, we obtain a better bound for $|\operatorname{Tail}(\phi, K)|$ in terms of $|S|$ and d.

Theorem 1.0.5. *Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S. Then*

$$|\operatorname{Tail}(\phi, K)| \leq d \max\left\{(5 \cdot 10^6(d^3+1))^{|S|+4}, 4(2^{64(|S|+3)})\right\}.$$

To get a similar bound for $|\operatorname{Per}(\phi, K)|$ we need to assume that ϕ has at least one K-rational tail point. Under this assumption, using Theorem 1.0.3 and results about Thue-Mahler equations, we can get:

Theorem 1.0.6. *Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S. If ϕ has at least one K-rational tail point then*

$$|\operatorname{Per}(\phi, K)| \leq \max\left\{(5 \cdot 10^6(d-1))^{|S|+3}, 4(2^{128(|S|+2)})\right\} + 1.$$

While the work described so far was being carried out, Canci and Vishkautsan [CV] proved a bound for $|\operatorname{Per}(\phi, K)|$, just assuming that ϕ has good reduction outside S. Their bound on $|\operatorname{Per}(\phi, K)|$ is roughly of the order of $d2^{16|S|} + 2^{2187|S|}$ where $d \geq 2$ is the degree of ϕ.

Now let's go through previous bounds for $|\operatorname{PrePer}(\phi, K)|$ which are relevant for our work. In 2007, Canci [Can07] proved for rational functions with good reduction outside S that the length of finite orbits is bounded by:

$$\left[e^{10^{12}}(|S|+1)^8(\log(5(|S|+1)))^8\right]^{|S|}. \tag{1.1}$$

Note that this bound depends only on the cardinality of S.

In Canci's recent work (2014) with Paladino [CP16] a sharper bound for the length of finite orbits was found:

$$\max\left\{(2^{16|S|-8}+3)\,[12|S|\log(5|S|)]^{[K:\mathbb{Q}]}, [12(|S|+2)\log(5|S|+5)]^{4[K:\mathbb{Q}]}\right\}. \tag{1.2}$$

In our work we are interested in the number of K-rational tail points and K-rational periodic points, $|\operatorname{Tail}(\phi, K)|$ and $|\operatorname{Per}(\phi, K)|$ respectively.

The bounds mentioned in (1.1) and (1.2) can be used to deduce bounds on $|\operatorname{PrePer}(\phi, K)|$. For instance, if we assume that every finite orbit has cardinality given by (1.1) and using that every point could have at most d preimages under ϕ we obtain a bound for $|\operatorname{PrePer}(\phi, K)|$ that is roughly of the order of $d^{(|S|\log|S|)^{8|S|}}$ where $d \geq 2$ is the degree of ϕ. Similarly, the bound deduced from (1.2) is roughly of the order of $d^{2^{16|S|}(|S|\log(|S|)^{[K:\mathbb{Q}]}}$, where $d \geq 2$ is the degree of ϕ. These bounds are polynomial in the degree of ϕ, however they will be rather large in terms of $|S|$.

In 2007, Benedetto [Ben07] proved for the case of polynomial maps of degree $d \geq 2$ that

$|\operatorname{PrePer}(\phi, K)|$ is bounded by $O(|S|\log|S|)$, where S is the set of places of K at which ϕ has bad reduction, including all archimedean places of K. The big-O is essentially $\frac{d^2-2d+2}{\log d}$ for large $|S|$.

Results in positive characteristic have also been found. For instance, in 2007 Ghioca [Ghi07] proved a bound for the number of torsion points of a Drinfeld module. In this case, torsion points are preperiodic points under the action of an additive polynomial of degree larger than one.

Another result in characteristic different from 0 is the work of Canci and Paladino [CP16] which gives a bound for the length of finite orbits under an endomorphism of $\mathbb{P}^1$.

The second part of this thesis provides quantitative and finiteness results for the set of K-rational tail curves of degree e for a given endomorphism of $\mathbb{P}^2$. Compared to the 1-dimensional case, a primary difficulty in proving higher-dimensional results comes from the limited availability of arithmetic tools in higher dimensions. Indeed, arithmetic tools used frequently in the one-dimensional setting include Siegel's theorem, Faltings' theorem, and Roth's theorem. Higher-dimensional conjectural generalizations of these results remain largely open, even for surfaces (e.g., Bombieri-Lang conjecture, Vojta's conjecture). A secondary difficulty comes from the more complicated geometry possible in higher dimensions. For instance, general position conditions (which appear, for example, in Vojta's conjecture) are rather trivial and uninteresting on curves. For these reasons, any progress towards the UBC in higher dimensions is highly valuable.

Even though the UBC in $\mathbb{P}^N$ is very hard there are some results on the literature. For instance, Hutz [Hut15] provides an algorithm to find $\mathbb{Q}$-rational preperiodic points for endomorphisms of $\mathbb{P}^n$. His techniques may be used to find a bound for the cardinality of the set of $\mathbb{Q}$-rational periodic points, depending on the smallest prime of good reduction.

Another important study in dimension bigger than one is the papers by J. Bell, D. Ghioca, and T. Tucker [BGT15], [BGT16]. In these papers we can find an example of infinitely many fixed curves for an endomorphism of $\mathbb{P}^2$. Indeed if f is a homogeneous two-variable polynomial of degree n, then the morphism $\mathbb{P}^2 \to \mathbb{P}^2$ given by $[x : y : z] \to [f(x,z) : f(y,z) : z^n]$ has infinitely many f-invariant curves of the form $[xz^{n^k-1} : f^k(x,z) : z^{n^k}]$, where f^k is the homogenized kth iterate of the dehomogenized one-variable polynomial $x \to f(x,1)$.

Motivated by the example of J. Bell, D. Ghioca, and T. Tucker and the Silverman-Morton Conjecture I study the set of K-rational preperiodic hypersurfaces of $\mathbb{P}^N$ under an endomorphism of $\mathbb{P}^N$. Let ϕ be an endomorphism of $\mathbb{P}^N$, defined over K, of degree d and H an irreducible K-rational hypersurface of $\mathbb{P}^N$ of degree e. We say that H is *periodic* under ϕ if there is an integer $n > 0$ such that $\phi^n(H) = H$. It is called *preperiodic* under ϕ if there is an integer $m \geq 0$ such that $\phi^m(H)$ is periodic. If H is preperiodic but not periodic it is called a *tail* hypersurface. Let $\mathrm{HTail}(\phi, K, e)$, $\mathrm{HPer}(\phi, K, e)$ and $\mathrm{HPrePer}(\phi, K, e)$ be the sets of K-rational tail, periodic and preperiodic hypersurfaces of degree e of ϕ, respectively.

It is important to notice that the degree of the preperiodic hypersurface will be involved in our study. This new parameter does not come up for points because the degree of a (geometric) point is always 1. However, this extra parameter is a natural condition because similar examples to the one given by J. Bell, D. Ghioca, and T. Tucker could be given if we consider subschemes in place of subvarieties. For instance, if instead of subvarieties we consider more generally integral closed K-subschemes, then a curve does have infinitely many periodic K-integral closed subschemes, because we can just take K-components of the subscheme of periodic points of period n. However, if we bound the degree of the K-subschemes, then once again we get finiteness by Northcott's theorem.

The main idea of my results on $\mathbb{P}^1$ [Tro] lies in an arithmetic relation between K-rational tail points and K-rational periodic points. Using a generalization of the $\mathfrak{p}$-adic logarithmic distance in $\mathbb{P}^1$, I was able to generalize the relation between K-rational tail points and K-rational periodic points to a relation between K-rational tail hypersurfaces and K-rational periodic points.

Theorem 1.0.7. *Let ϕ be an endomorphism of $\mathbb{P}^n$, defined over K. Suppose ϕ has good reduction outside S. Let H be a K-rational tail hypersurface, m the period of the periodic part of the orbit of H and H' the periodic hypersurface such that $H' = \phi^{m_0 m}(H)$ for some $m_0 > 0$. Let $P \in \mathbb{P}^n(K)$ be any periodic point such that $P \notin supp\{H'\}$. Then $\delta_v(P; H) = 0$ for every $v \notin S$.*

In [GTZ11] Bell, Ghioca and Tucker also propose the following question

Question: *Is there a constant $C = C(N, K, d)$ such that for any periodic K-rational subvariety V of $\mathbb{P}^N$, we have $\mathrm{Per}_\Phi(V) \leq C$?*

Using the previous arithmetic relation together with a result from Ru and Wong [RW91] we give a result that implies a partial answer to the previous question for curves on the projective plane. In fact, we provide a bound for the number of K-rational tail hypersurfaces of degree e in the backwards orbit of a given periodic K-rational hypersurface of $\mathbb{P}^n$.

Theorem 1.0.8. *Let ϕ be an endomorphism of $\mathbb{P}^n$, defined over K and suppose ϕ has good reduction outside S. Consider $N = \binom{e+n}{e} - 1$ and let $\{P_i\}_{i=1}^{2N+1}$ be a set of K-rational periodic points of $\mathbb{P}^n$ such that no $N+1$ of them lie in a curve of degree e. Consider $\mathcal{B} = \{H' \in \mathrm{HPer}(\phi, K) \colon \forall 1 \leq i \leq 2N+1, \quad P_i \notin supp\ H'\}$ and $\mathcal{A} = \{q \in \mathrm{HTail}(\phi, K, e) \colon$ there is $H' \in \mathcal{B}$ and $l \geq 0, \quad \phi^{l n_q}(q) = H'$ where n_q is the period of the periodic part of $q\}$. Then*

$$|\mathcal{A}| \leq \left(2^{33} \cdot (2N+1)^2\right)^{(N+1)^3 (s+2N+1)}$$

In 2016 B. Hutz [Hut16] proved that the set of K-rational preperiodic subvarieties of $\mathbb{P}^n$ is finite. His proof is based on the theory of canonical height functions. In the special case of K-rational preperiodic curves of $\mathbb{P}^2$ we were able to give an alternative proof than the one given by Hutz. This alternative proof is based in a strong result of dynamical systems ([Fak03], Corollary 5.2) which states that if ϕ is an endomorphism of $\mathbb{P}^n$ then the set $\mathrm{Per}(\phi, \bar{K})$ for an endomorphism ϕ is Zariski dense in $\mathbb{P}^n$.

Theorem 1.0.9. *Let K be a number field and ϕ be an endomorphism of $\mathbb{P}^2$, defined over K. Then for every $e \in \mathbb{N}$ the set $\mathrm{HTail}(\phi, K, e)$ is finite.*

We end this introduction with a brief outline of the rest of the thesis. **??** introduces some classical notations and definitions from arithmetic dynamics, arithmetic geometry and number theory. We also prove some propositions needed for the main theorems of this manuscript.

?? presents the proof of our results on $\mathbb{P}^1$. This chapter has three sections: the first section gives all the propositions and lemmas needed for the next two sections, the second section uses S-unit equations to get bounds for the set of K-rational preperiodic points and the third section uses Thue-Mahler equations to gives different bounds for the set of K-rational preperiodic points.

Finally, **??** presents definitions and results on $\mathbb{P}^N$. This chapter has four sections: the first one gives definitions and propositions on $\mathbb{P}^N$. The second section give effective results for a large subset of the set of K-rational tail hypersurfaces of $\mathbb{P}^N$ of a given degree. The third section prove finiteness of the set of K-rational tail curves of degree e of $\mathbb{P}^2$. The last section gives examples of K-rational tail and periodic hypersurfaces of $\mathbb{P}^N$.

BIBLIOGRAPHY

BIBLIOGRAPHY

[Bak64] I. Baker. Fixpoints of polynomials and rational functions. *J. London Math. Soc.*, 39:615–622, 1964.

[BCH$^+$14] R. Benedetto, R. Chen, T. Hyde, Y. Kovacheva, and C. White. Small dynamical heights for quadratic polynomials and rational functions. *Exp. Math.*, 23(4):433–447, 2014.

[Ben07] R. Benedetto. Preperiodic points of polynomials over global fields. *J. Reine Angew. Math.*, 608:123–153, 2007.

[BGT15] J. Bell, D. Ghioca, and T. Tucker. Applications of p-adic analysis for bounding periods for subvarieties under étale maps. *Int. Math. Res. Not. IMRN*, (11):3576–3597, 2015.

[BGT16] J. Bell, D. Ghioca, and T. Tucker. *The dynamical Mordell-Lang conjecture*, volume 210 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2016.

[BS96] F. Beukers and H. Schlickewei. The equation $x + y = 1$ in finitely generated groups. *Acta Arith.*, 78(2):189–199, 1996.

[Can07] J. Canci. Finite orbits for rational functions. *Indag. Math. (N.S.)*, 18(2):203–214, 2007.

[Can10] J. Canci. Rational periodic points for quadratic maps. *Ann. Inst. Fourier (Grenoble)*, 60(3):953–985, 2010.

[CP16] J. Canci and L. Paladino. Preperiodic points for rational functions defined over a global field in terms of good reduction. *Proc. Amer. Math. Soc.*, 144(12):5141–5158, 2016.

[CV] J. Canci and S. Vishkautsan. Scarcity of cycles for rational functions over a number field. *Trans. Amer. Math. Soc. Ser. B*, to appear.

[Eve95] J. Evertse. The number of solutions of decomposable form equations. *Invent. Math.*, 122(3):559–601, 1995.

[Eve97] J. Evertse. The number of solutions of the Thue-Mahler equation. *J. Reine Angew. Math.*, 482:121–149, 1997.

[Fak01] N. Fakhruddin. Boundedness results for periodic points on algebraic varieties. *Proc. Indian Acad. Sci. Math. Sci.*, 111(2):173–178, 2001.

[Fak03] N. Fakhruddin. Questions on self maps of algebraic varieties. *J. Ramanujan Math. Soc.*, 18(2):109–122, 2003.

[FHI$^+$09] X. Faber, B. Hutz, P. Ingram, R. Jones, M. Manes, T. Tucker, and M. Zieve. Uniform bounds on pre-images under quadratic dynamical systems. *Math. Res. Lett.*, 16(1):87–101, 2009.

[Ghi07] D. Ghioca. The Lehmer inequality and the Mordell-Weil theorem for Drinfeld modules. *J. Number Theory*, 122(1):37–68, 2007.

[GTZ11] D. Ghioca, T. Tucker, and S. Zhang. Towards a dynamical Manin-Mumford conjecture. *Int. Math. Res. Not. IMRN*, (22):5109–5122, 2011.

[HI13] B. Hutz and P. Ingram. On Poonen's conjecture concerning rational preperiodic points of quadratic maps. *Rocky Mountain J. Math.*, 43(1):193–204, 2013.

[Hut15] B. Hutz. Determination of all rational preperiodic points for morphisms of PN. *Math. Comp.*, 84(291):289–308, 2015.

[Hut16] B. Hutz. Good reduction and canonical heights of subvarieties. *ArXiv e-prints*, March 2016.

[Kis95] M. Kisaka. On some exceptional rational maps. *Proc. Japan Acad. Ser. A Math. Sci.*, 71(2):35–38, 1995.

[Man07] M. Manes. *Arithmetic dynamics of rational maps.* ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)–Brown University.

[Maz77] B. Mazur. Modular curves and the Eisenstein ideal. *Inst. Hautes Études Sci. Publ. Math.*, (47):33–186 (1978), 1977.

[Mer96] L. Merel. Bornes pour la torsion des courbes elliptiques sur les corps de nombres. *Invent. Math.*, 124(1-3):437–449, 1996.

[MN06] R. Marszałek and W. Narkiewicz. Finite polynomial orbits in quadratic rings. *Ramanujan J.*, 12(1):91–130, 2006.

[MS94] P. Morton and J. Silverman. Rational periodic points of rational functions. *Internat. Math. Res. Notices*, (2):97–110, 1994.

[Nor50] D. Northcott. Periodic points on an algebraic variety. *Ann. of Math. (2)*, 51:167–177, 1950.

[NW02] J. Noguchi and J. Winkelmann. Holomorphic curves and integral points off divisors. *Math. Z.*, 239(3):593–610, 2002.

[Poo98] B. Poonen. The classification of rational preperiodic points of quadratic polynomials over $\mathbf{Q}$: a refined conjecture. *Math. Z.*, 228(1):11–29, 1998.

[Rib01] P. Ribenboim. *Classical theory of algebraic numbers*. Universitext. Springer-Verlag, New York, 2001.

[RW91] M. Ru and P. Wong. Integral points of $\mathbf{P}^n - \{2n + 1 \text{ hyperplanes in general position}\}$. *Invent. Math.*, 106(1):195–216, 1991.

[Sil07] J. Silverman. *The arithmetic of dynamical systems*, volume 241 of *Graduate Texts in Mathematics*. Springer, New York, 2007.

[Tro] S. Troncoso. Bound for preperiodic points for maps with good reduction. *J. Number Theory*, to appear.

[Voj87] P. Vojta. *Diophantine approximations and value distribution theory*, volume 1239 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.