BASIC TOPOLOGICAL FACTS

TOPOLOGIES

Definition: A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset, X \in \mathcal{T}$
2. $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$
3. $U_\alpha \in \mathcal{T}, \forall \alpha \in \Lambda \Rightarrow \bigcup_{\alpha \in \Lambda} U_\alpha \in \mathcal{T}$

Definition: A **topological space** $(X, \mathcal{T})$ is an ordered pair where X is a set and $\mathcal{T}$ is a topology on X.

Definition: If $U \in \mathcal{T}$, we say that U is an **open** subset of X.

Proposition: If $\{U_i\}_{i=1}^n \subseteq \mathcal{T}$ for some n, then $\bigcap_{i=1}^n \in \mathcal{T}$.

Proposition: $U \in \mathcal{T} \Leftrightarrow \forall x \in U, \exists V \in \mathcal{T}$ such that $x \in V \subseteq U$.

(Thus U is an open subset of X iff for each $x \in U$,
there exists an **open neighborhood** V of x such that $V \subseteq U$.)

BASES

Definition: If $(X, \mathcal{T})$ is a topological space, a **basis** for $\mathcal{T}$ is a subcollection $\mathcal{B}$ of $\mathcal{T}$ with the property that if $U \in \mathcal{T}$ then there is a subcollection $\mathcal{B}'$ (possibly empty) of $\mathcal{B}$ such that $U = \bigcup_{B \in \mathcal{B}'} B$.

Proposition: $\mathcal{B}$ is a basis for $\mathcal{T}$ iff $\forall x \in U$, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Proposition: Given a set X, let $\mathcal{B}$ be a collection of subsets of X satisfying the properties:

1. $X = \bigcup_{B \in \mathcal{B}} B$

2. $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2 \Rightarrow \exists B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$

Then the collection of all unions (possibly empty) of elements of $\mathcal{B}$ forms a topology $\mathcal{T}$ on X and $\mathcal{B}$ is a basis for $\mathcal{T}$.

SUBBASES

Definition: If $(X, \mathcal{T})$ is a topological space, a **subbasis** for $\mathcal{T}$ is a subcollection $\mathcal{L}$ of $\mathcal{T}$ such that the collection of all finite intersections (possibly empty) of elements of $\mathcal{L}$ is a basis for $\mathcal{T}$.

Proposition: Given a set X, let $\mathcal{L}$ be a collection of subsets of X. Then the collection of all unions (possibly empty) of all finite intersections (possibly empty) of elements of $\mathcal{L}$ forms a topology on X and $\mathcal{L}$ is a subbasis for $\mathcal{T}$.

SUBSPACES AND PRODUCT SPACES

Proposition: Let $(X, \mathcal{T})$ be a topological space. Let $A \subseteq X$.
Let $\mathcal{T}_A = \{B \subseteq A : B = A \cap U \text{ for some } U \in \mathcal{T}\}$. Then $\mathcal{T}_A$ is a topology on A.

Definition: Let $(X, \mathcal{T})$ be a topological space. Let $A \subseteq X$.
Then $\mathcal{T}_A$ is called the **subspace topology** on A, and $(A, \mathcal{T}_A)$ is called a **topological subspace** of $(X, \mathcal{T})$.

Remark: Let $(A, \mathcal{T}_A)$ be a topological subspace of a topological space $(X, \mathcal{T})$.
B is an open subset of A iff $B = A \cap U$ for some open subset U of X.

Proposition: Let $\{(X_i, \mathcal{T}_i)\}_{i=1}^n$ be a set of topological spaces. Let $\mathcal{B} = \{U_1 \times \cdots \times U_n : U_i \in \mathcal{T}_i, \forall i = 1, \cdots, n\}$.
Then $\mathcal{B}$ is a basis for some topology $\mathcal{T}$.

Definition: Let $\{(X_i, \mathcal{T}_i)\}_{i=1}^n$ be a set of topological spaces. The topology $\mathcal{T}$ generated by $\mathcal{B}$ in the previous proposition is called the **product topology** on $X_1 \times \cdots \times X_n$, and $(X_1 \times \cdots \times X_n, \mathcal{T})$ is called a **product space**.

MAPS

Definition: Let $f : X \to Y$ and let $A \subseteq X$. A is said to be **saturated** if $A = f^{-1}(f(A))$.

Proposition: If $f : X \to Y$ is injective, and $A \subseteq X$, then A is saturated.

Definition: Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces.
A map $f : X \to Y$ is said to be **continuous** if $V \in \mathcal{T}_2 \Rightarrow f^{-1}(V) \in \mathcal{T}_1$
(thus for every open subset V of Y, $f^{-1}(V)$ is an open subset of X.)

Proposition: Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces.
If $f : X \to Y$ is continuous, $A \subseteq X$, and $f(X) \subseteq B \subseteq Y$ then $f|_A^B : A \to B$ is continuous.

Definition: Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces.
A map $f : X \to Y$ is said to be **open** if $U \in \mathcal{T}_1 \Rightarrow f(U) \in \mathcal{T}_2$
(thus for every open subset U of X, $f(U)$ is an open subset of Y.)

Definition: Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. A map $f : X \to Y$ is said to be a **homeomorphism** if

1. f is bijective

2. f **and** f^{-1} are continuous.

Proposition: Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. Let $f : X \to Y$ be a bijective continuous map.
Then $f : X \to Y$ is a homeomorphism iff $f : X \to Y$ is an open map.

QUOTIENT MAPS

Definition: Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. A map $f : X \to Y$ is said to be a **quotient map** if

1. f is surjective

2. f is continuous

3. If $B \subseteq Y$ such that $f^{-1}(B)$ is an open subset of X, then B is an open subset of Y (i.e. $f^{-1}(B) \in \mathcal{T}_1 \;\Rightarrow\; B \in \mathcal{T}_2$)

Remark: Conditions 2 and 3 above are sometimes written together as
"B is an open subset of Y iff $f^{-1}(B)$ is an open subset of X"

Important Point: Condition 3 does **NOT** imply that $f : X \to Y$ is an open map.
Thus we can **NOT** say that A an open subset of $X \;\Rightarrow\; f(A)$ is an open subset of Y.
The problem is that for any generic subset $A \subseteq X$, A is not necessarily saturated.
In general, quotient maps need not be injective.

Proposition: If $f : X \to Y$ is a quotient map, f takes **saturated** open sets to open sets.

Proposition: Let $(X, \mathcal{T}_1)$ be a topological space. Let $f : X \to Y$ be a surjective map.
Let $\mathcal{T}_2 = \{U \subseteq Y : f^{-1}(U) \in \mathcal{T}_1\}$. Then $\mathcal{T}_2$ is a topology on X and $f : X \to Y$ is a quotient map.

Definition: The topology $\mathcal{T}_2$ constructed in the above proposition is called the **quotient topology** on Y induced by f.

Proposition: Let $(X, \mathcal{T}_1)$ be a topological space. Let $\sim$ be an equivalence relation on X.
Let $X/\!\sim$ denote the set of equivalence classes defined by $\sim$.
Let $\pi : X \to X/\!\sim$ denote the natural projection $\pi(x) = [x]$.
Then $\pi : X \to X/\!\sim$ is surjective and letting $\mathcal{T}_2$ be the quotient topology on $X/\!\sim$ induced by π,
$\pi : X \to X/\!\sim$ is a quotient map.

Definition: Let $(X, \mathcal{T}_1)$ be a topological space, and let $\sim$ be an equivalence relation on X.
If $\mathcal{T}_2$ is the quotient topology on $X/\!\sim$, $(X/\!\sim, \mathcal{T}_2)$ is called the **quotient space** of $(X, \mathcal{T}_1)$ with respect to $\sim$.

Proposition: If $f : X \to Y$ is a quotient map, and $A \subseteq X$ is a **saturated open** set,
then $f|_A : A \to Y$ and $f|_A^{f(A)} : A \to f(A)$ are quotient maps.

Proposition: If $f : X \to Y$ is a quotient map and U is an open subset of Y,
then $f|_{f^{-1}(U)}^{U} : f^{-1}(U) \to U$ is a quotient map.

Proposition: If $f : X \to Y$ is a homeomorphism, f is a quotient map.

Proposition: If $f_1 : X \to Y$ is a quotient map and $f_2 : Y \to Z$ is a quotient map, then $f_2 \circ f_1 : X \to Z$ is a quotient map.

Proposition (Universal Property of Quotient Maps): Let $(X, \mathcal{T}_1)$, $(Y, \mathcal{T}_2)$, and $(Z, \mathcal{T}_3)$ be topological spaces. Let $\pi : X \to Y$ be a quotient map. Then $f : Y \to Z$ is continuous iff $f \circ \pi : X \to Z$ is continuous.

SECOND COUNTABLE, HAUSDORFF, LOCALLY EUCLIDEAN

Definition: A topological space $(X, \mathcal{T})$ is called **second countable** if $\mathcal{T}$ has a countable basis.

Definition: A topological space $(X, \mathcal{T})$ is called **Hausdorff** if
$\forall p_1, p_2 \in X$ with $p_1 \neq p_2$, $\exists U_1, U_2 \in \mathcal{T}$ (i.e. U_1, U_2 are open subsets of X) such that
$p_1 \in U_1$, $p_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.

Definition: A topological space $(X, \mathcal{T})$ is called **locally Euclidean** if
$\forall p \in X$, $\exists U \in \mathcal{T}$ (i.e. U is an open subset of X) with $p \in U$, $\exists V \subseteq \mathbb{R}^n$ with V an open subset of $\mathbb{R}^n$,
and $\exists$ a homeomorphism $\varphi : U \to V$.

Proposition: Any subspace of a Hausdorff space is Hausdorff and any product of Hausdorff spaces is Hausdorff.

Proposition: Any subspace of a second countable space is second countable and any product of second countable spaces is second countable.

Proposition (Quotient Space Result): Let $f : X \to Y$ be a quotient map.
If X is second countable and Y is locally Euclidean, then Y is second countable.