MORE ON THE TENSOR PRODUCT

Steven Sy

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3.1 Commutative Rings

A. Introduction

Let R be a **commutative** ring (with 1).

Let M_1 and M_2 be R-modules.

Then the abelian group $M_1 \otimes_R M_2$ is an R**-module** under scalar multiplication defined by $r(a \otimes b) = (ra) \otimes b$.

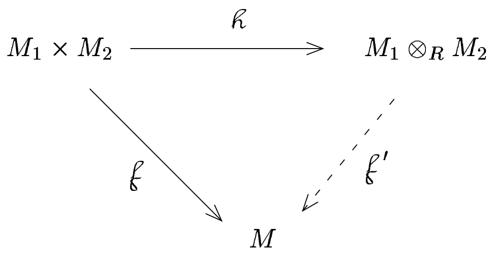
Definition: Let M_1, M_2, M be R-modules.

Then $f : M_1 \times M_2 \to M$ is called an R**-bilinear** function if f satisfies the following properties:

1. f is R-biadditive
2. $rf(a, b) = f(ra, b) = f(a, rb)$ $\quad \forall r \in R,\ a \in M_1,\ b \in M_2$

B. Universal Property of $M_1 \otimes_R M_2$ (Module Version)

For every R-module M and every R-bilinear function $\mathfrak{f} : M_1 \times M_2 \to M$, there exists a unique R-map $\mathfrak{f}' : M_1 \otimes_R M_2 \to M$ such that the following diagram commutes:



Proof

Given M and $\mathfrak{f}$ as above, M is an abelian group and $\mathfrak{f}$ is R-biadditive, so by the Universal Property, there exists a unique **homomorphism** $\mathfrak{f}' : M_1 \otimes M_2 \to M$ so that the diagram commutes.

Then

$$
\begin{aligned}
\mathfrak{f}'(r(a \otimes b)) &= \mathfrak{f}'((ra) \otimes b) && \text{[definition of } M_1 \otimes_R M_2 \text{ module structure]} \\
&= \mathfrak{f}'\mathfrak{k}(ra, b) \\
&= \mathfrak{f}(ra, b) && \text{[commutative diagram]} \\
&= r\mathfrak{f}(a, b) && [\mathfrak{f} \text{ is bilinear}] \\
&= r\mathfrak{f}'\mathfrak{k}(a, b) && \text{[commutative diagram]} \\
&= r\mathfrak{f}'(a \otimes b)
\end{aligned}
$$

Similarly, for the 2nd component.

C. Uniqueness

Let X be an R-module with the property:

There exists an R-bilinear function $\mathfrak{k} : M_1 \times M_2 \to X$ such that:

For every R-module M and every R-bilinear function $\mathfrak{f} : M_1 \times M_2 \to M$, there exists a unique R-map $\mathfrak{f}' : X \to M$.

Then $X \cong M_1 \otimes_R M_2$ (as R-modules).

Proof

Similar to before; Exercise

3.2 Ring Tensoring

A. Basic Facts

1. $\boxed{R \otimes_R M \cong M}$ via $r \otimes a \overset{\Phi}{\mapsto} ra$ on generators

Proof

Define $\varphi : R \times M \to M$ by $\varphi(r, a) = ra$.

Then φ is R-bilinear, so by the Universal Property (Module Version),
φ extends to an R-map $\Phi : R \otimes_R M \to M$ such that $\Phi(r \otimes a) = ra$.

Define $\Psi : M \to R \otimes_R M$ by $a \mapsto 1 \otimes a$

Then Ψ is an R-map.

Furthermore, $\Psi\Phi(r \otimes a) = \Psi(ra) = 1 \otimes (ra) = r \otimes a$

Then $\Psi\Phi = id_{R \otimes_R M}$.

Similarly, $\Phi\Psi = id_M$, so Φ is an isomorphism with inverse Ψ.

2. $\boxed{M \otimes_R R \cong M}$

Proof

$a \otimes r \mapsto ar$ defines an isomorphism as in #1 with inverse $a \mapsto a \otimes 1$

B. Base Change

Let M be an R-module
Let S be an R-algebra (i.e. S is an R-module and S is a ring)
Then $S \otimes_R M$ is an S-module.

Proof

Since S is an R-module, $S \otimes_R M$ is an abelian group.
Then $\forall s \in S$, define $s(a \otimes b) = (sa) \otimes b$ on generators, to make $S \otimes_R M$ an S-module.

C. Examples

1. $\boxed{\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Q}}$ and $\boxed{\left(\mathbb{Z}/p\mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Z} = \left(\mathbb{Z}/p\mathbb{Z}\right)}$ (p a prime)

Proof

By Section B.

2. $\boxed{\left(\mathbb{Z}/p\mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Q} = 0}$ (p a prime)

Proof

Let $a \otimes b$ be a generator in $\left(\mathbb{Z}/p\mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Then $a \otimes b = (pa) \otimes \frac{b}{p} = \bar{0} \otimes \frac{b}{p} = 0.$

3. $\boxed{\left(\mathbb{Q}/\mathbb{Z}\right) \otimes_{\mathbb{Z}} \left(\mathbb{Q}/\mathbb{Z}\right) = 0}$

Proof

Let $\bar{a} \otimes \bar{b}$ be a generator in $\left(\mathbb{Q}/\mathbb{Z}\right) \otimes_{\mathbb{Z}} \left(\mathbb{Q}/\mathbb{Z}\right)$.

Then for some $s \neq 0$, $\bar{a} = \frac{r}{s} + \mathbb{Z}$.

Then

$$\begin{aligned}
\bar{a} \otimes \bar{b} &= \left(\frac{r}{s} + \mathbb{Z}\right) \otimes s\left(\frac{1}{s}\,\bar{b}\right) \\
&= s\left(\frac{r}{s} + \mathbb{Z}\right) \otimes \left(\frac{1}{s}\,\bar{b}\right) \\
&= (r + \mathbb{Z}) \otimes \left(\frac{1}{s}\,\bar{b}\right) \\
&= \bar{0} \otimes \left(\frac{1}{s}\,\bar{b}\right) \\
&= 0
\end{aligned}$$

4. If V is an $\mathbb{R}$-vector space, then $V \otimes_R \mathbb{C}$ is a $\mathbb{C}$-vector space.

Proof

By Section B

3.3 Isomorphism Theorems

A. Commutativity

$\boxed{M_1 \otimes_R M_2 \cong M_2 \otimes_R M_1}$ via $a \otimes b \overset{\Phi_C}{\mapsto} b \otimes a$ on generators

Proof

Let $\ell : M_1 \times M_2 \to M_2 \otimes_R M_1,$ be defined by $\ell(a,b) = b \otimes a.$

Since ℓ is R-bilinear, by the Universal Property (Module Version),
it extends to a unique R-map $\Phi_C : M_1 \otimes_R M_2 \to M_2 \otimes M_1$ with $a \otimes b \mapsto b \otimes a.$

We similarly get an R-map $\Psi_C : M_2 \otimes_R M_1 \to M_1 \otimes M_2$ with $b \otimes a \mapsto a \otimes b.$

Then Φ_C and Ψ_C are inverses, so Φ_C is an isomorphism.

B. Associativity

$\boxed{(M_1 \otimes_R M_2) \otimes_R M_3 \cong M_1 \otimes_R (M_2 \otimes_R M_3)}$ via $(a \otimes b) \otimes c \overset{\Phi_A}{\mapsto} a \otimes (b \otimes c)$ on generators

Proof

$\forall c \in M_3$, define $\ell_c : M_1 \times M_2 \to M_1 \otimes_R (M_2 \otimes_R M_3)$ by $(a,b) \mapsto a \otimes (b \otimes c)$

Then ℓ_c is R-bilinear, so by the Universal Property (Module Version),
it extends to a unique R-map $\ell_c' : M_1 \otimes_R M_2 \to M_1 \otimes_R (M_2 \otimes_R M_3)$
where $a \otimes b \mapsto a \otimes (b \otimes c)$

Define $\ell : (M_1 \otimes_R M_2) \times M_3 \to M_1 \otimes_R (M_2 \otimes_R M_3)$ by $\ell(\alpha, c) = \ell_c'(\alpha).$

Since ℓ_c' is an R-map, $\forall c \in C,$ ℓ is R-bilinear.

Thus, by the Universal Property (Module Version),
ℓ extends to a unique R-map $\Phi_A : (M_1 \otimes_R M_2) \otimes_R M_3 \to M_1 \otimes_R (M_2 \otimes_R M_3)$
where $(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$

The inverse is constructed similarly.

3.4 The Multi-Tensor Product

A. R k-Multilinear Functions

Let $M_1, \dots, M_k, M$ be R-modules.

Then $f : M_1 \times \cdots \times M_k \to M$ is called an **R k-multilinear function** if the following holds:

1. $f(a_1, \dots, a_i + a_i', \dots, a_k) = f(a_1, \dots, a_i, \dots, a_k) + f(a_1, \dots, a_i', \dots, a_k)$

2. $f(a_1, \dots, ra_i, \dots, a_k) = r f(a_1, \dots, a_i, \dots, a_k)$

$$\forall 1 \le i \le k, \quad \forall a_i, a_i' \in M_i, \quad \forall r \in R$$

B. The k-Multi-Tensor Product

Given R-modules $M_1, \dots, M_k, M$, we define

$$M_1 \otimes M_2 \otimes \cdots \otimes M_k = F_R < M_1 \times \cdots \times M_k > \big/ V$$

where V is the R-submodule of $F_R < M_1 \times \cdots \times M_k >$ generated by the elements:

1. $(a_1, \dots, a_i + a_i', \dots, a_k) - (a_1, \dots, a_i, \dots, a_k) - (a_1, \dots, a_i', \dots, a_k)$

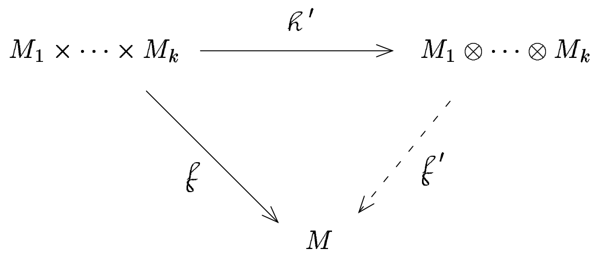
2. $(a_1, \dots, ra_i, \dots, a_k) - r f(a_1, \dots, a_i, \dots, a_k)$

$$\forall 1 \le i \le k, \quad \forall a_i, a_i' \in M_i, \quad \forall r \in R$$

Definition: Let $k' : M_1 \times \cdots \times M_k \to M_1 \otimes \cdots \otimes M_k$ be the canonical map
where $(a_1, \dots, a_k) \mapsto (a_1, \dots, a_k) + V$

C. Universal Property of the k-Multi-Tensor Product

For every R-module M and every R k-multilinear function $\mathscr{f} : M_1 \times \cdots \times M_k \to M$, there exists a unique R-map $\mathscr{f}' : M_1 \otimes \cdots \otimes M_k \to M$ such that the following diagram commutes:



Proof

Similar to before (Exercise)

D. Uniqueness

Let X be an R-module with the property:

There exists an R k-multilinear function $\mathscr{k} : M_1 \times \cdots \times M_k \to X$ such that:

For every R-module M and every R k-multilinear function $\mathscr{f} : M_1 \times \cdots \times M_k \to M$, there exists a unique R-map $\mathscr{f}' : X \to M$.

Then $X \cong M_1 \otimes \cdots \otimes M_k$

Proof

Similar to before (Exercise)

E. Comments

1. For M_1, M_2 R-modules, $M_1 \otimes M_2 \cong M_1 \otimes_R M_2$

   Proof

   Both satisfy the Universal Property of the R 2-Multi-Tensor Product.
   Do a uniqueness argument.

2. In view of #1, we write $a \otimes b$ for the generators in both (isomorphic) modules, as an "abuse of notation".

3. For M_1, M_2, M_3 R-modules,

$$M_1 \otimes M_2 \otimes M_3 \;\cong\; (M_1 \otimes_R M_2) \otimes_R M_3 \;\cong\; M_1 \otimes_R (M_2 \otimes_R M_3)$$

4. Similar results as above hold for k R-modules $M_1, \ldots, M_k$.

5. **Notation:** $a_1 \otimes \cdots \otimes a_k = (a_1, \ldots, a_k) + V$

6. All of these k-multitensor products in this Chapter can be defined appropriately in general (**noncommutative**) rings using **bimodules**

   i.e. $(A_R) \otimes_R ({}_R B_S) \otimes_S ({}_S C)$ etc.