INTRODUCTION TO THE TENSOR PRODUCT

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2.1 Preliminaries

A. Formal Linear Combinations: Intuition

Given a set X, we wish to give meaning to "$3x_1 + 2x_2$" and make the collection of such objects an R-module.

B. Construction

Let X be a set and let R be a ring (with 1). To each $x \in X$, define $\propto : X \to R$ by $\propto(y) = \begin{cases} 1; & \text{if } y = x \\ 0; & \text{if } y \neq x \end{cases}$.

(Thus $\propto$ is a "characteristic function" with value 1 at x.)

Definition: For all $x_j \in X$, define $1 \cdot x_j = x_j$.

For all $x_i \in X$ and $c_i \in R$, if the sum $\sum_{i=1}^{n} c_i x_i$ is not already x_j, define it to be $\sum_{i=1}^{n} c_i \propto_i$.

$\sum_{i=1}^{n} c_i x_i$ is called a **formal linear combination** of elements of X.

Definition: Let $F_R<X>$ be the set of all formal linear combinations of elements of X.

Then under pointwise addition of functions, and scalar multiplication by elements of R,

$F_R<X>$ is an R-module.

C. Comments

1. $F_R{<}X{>}$ is a free R-module with basis X.

2. $X \subseteq F_R{<}X{>}$

3. For $R = \mathbb{Z}$, we call $F_{\mathbb{Z}}{<}X{>}$ the free abelian group on X.

4. By the construction, given any set X, there exists a free R-module/abelian group having X as a basis.

D. Notation

1. If A is a left R-module, we write ${}_RA$.

2. If A is a right R-module, we write A_R.

E. R-biadditive functions

Definition: f is called an R-**biadditive function** if

$$f : (A_R) \times ({}_RB) \to G$$

where G is an abelian group.

Furthermore f must satisfy the following properties:

1. $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b) \qquad \forall a_1, a_2 \in A;\ b \in B$

2. $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2) \qquad \forall a \in A;\ b_1, b_2 \in B$

3. $f(ar, b) = f(a, rb) \qquad \forall a \in A,\ b \in B,\ r \in R$

F. Tensor Product Intuition

We wish to construct a special type of “product” that “behaves biadditively”.

Now formalize . . .

2.2 The Tensor Product

A. Definition

Given A_R and ${}_RB$, we define $A \otimes_R B$:

$$A \otimes_R B = {F_{\mathbb{Z}}\langle A \times B\rangle} \big/ {U}$$

where U is the subgroup of $F_{\mathbb{Z}}\langle A \times B\rangle$ which is generated by the elements:

1. $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$

2. $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$

3. $(ar, b) - (a, rb)$

where $a, a_1, a_2 \in A;\ b, b_1, b_2 \in B;\ r \in R$.

B. Notation and Comments

1. $a \otimes b = (a, b) + U \in A \otimes_R B$

2. Since elements of $F_{\mathbb{Z}}\langle A \times B\rangle$ look like $\sum_{i=1}^{n} c_i(a_i, b_i)$ for $c_i \in \mathbb{Z}$, elements of $A \otimes_R B$ look like

$$\sum_{i=1}^{n} c_i(a_i, b_i) + U = \sum_{i=1}^{n} c_i(a_i \otimes b_i)$$

3. Since we may choose different representatives from a coset, the representation $\sum_{i=1}^{n} c_i(a_i \otimes b_i)$ for a typical element in $A \otimes_R B$ is **not** unique.

4. The **zero element** in $A \otimes_R B$, is $0 \otimes 0 = (0, 0) + U$, which is sometimes simply written as 0.

5. **Properties:**

   a. $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$

   b. $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$

   c. $(ar) \otimes b = a \otimes (rb)$

(These are all true by definition of U.)

6. **Additional Properties:**

   a. $a \otimes 0 = 0$

      Proof

      $$a \otimes 0 + a \otimes 0 \overset{\text{Prop (b)}}{=} a \otimes 0$$

      Thus, $a \otimes 0 = 0$.

   b. $0 \otimes b = 0$

7. Since elements of $A \otimes_R B$ are cosets, when you define a function with domain $A \otimes_R B$ you must make sure that it is well-defined. By definition of U, this amounts to showing that the proposed function is R-biadditive on the underlying Cartesian product.

8. **Warning:** If $A \subseteq B$, it is **not** true in general that $A \otimes_R C \subseteq B \otimes_R C$!
   Reason: something involving "flatness" (See later)

9. Let $\mathscr{k} : A \times B \to A \otimes_R B$ be the canonical map defined by

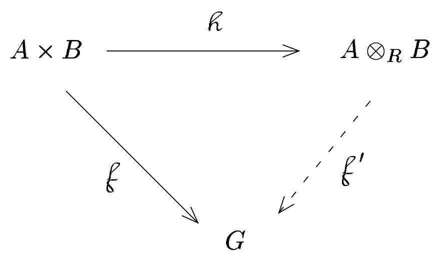
$$(a_i, b_i) \overset{\mathscr{k}}{\mapsto} (a_i, b_i) + U = a_i \otimes b_i$$

   Note that $\mathscr{k}$ is R-biadditive.

10. The tensor product completely characterizes R-biadditive maps.
    We will see this in the **Universal Property**.

C. Universal Property of $A \otimes_R B$

For every abelian group G and every R-biadditive function $f : A \times B \to G$, there exists a unique homomorphism $f' : A \otimes_R B \to G$ such that the following diagram commutes:



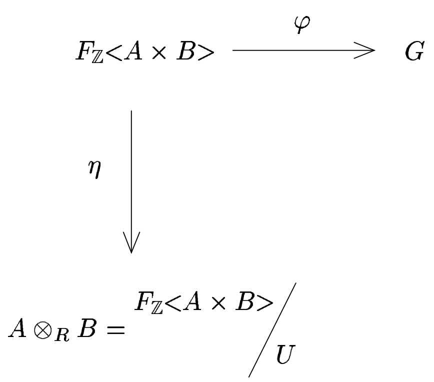
Proof

Given an abelian group G and R-biadditive function $f : A \times B \to G$, there exists a unique homomorphism $\varphi : F_{\mathbb{Z}}<A \times B> \to G$ extending f, namely

$$\varphi\left(\sum_{i=1}^{n} c_i(a_i, b_i)\right) = \sum_{i=1}^{n} c_i f(a_i, b_i)$$

Let $\eta : F_{\mathbb{Z}}<A \times B> \longrightarrow F_{\mathbb{Z}}<A \times B>\big/U$ be the canonical map.

Then we have the following diagram:



Since f is R-biadditive, $U \subseteq \ker \varphi$, so φ factors uniquely through η to give a homomorphism $f' : A \otimes_R B \to G$ defined by

$$\sum_{i=1}^{n} c_i(a_i \otimes b_i) \overset{\mathscr{f}'}{\mapsto} c_i \mathscr{f}(a_i, b_i)$$

Then $\mathscr{f}'\mathscr{h}(a, b) = \mathscr{f}'(a \otimes b) = \mathscr{f}(a, b)$.

Thus $\mathscr{f}'\mathscr{h} = \mathscr{f}$, and the diagram commutes.

D. Uniqueness

Let X be an abelian group with the property:

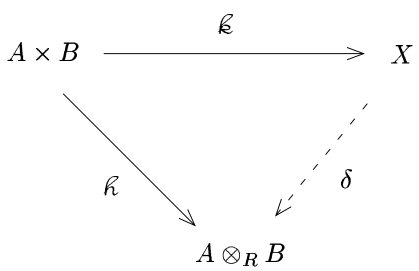
There exists an R-biadditive function $\mathscr{k} : A \times B \to X$ such that:

For every abelian group G and every R-biadditive function $\mathscr{f} : A \times B \to G$, there exists a unique homomorphism $\mathscr{f}' : X \to G$.

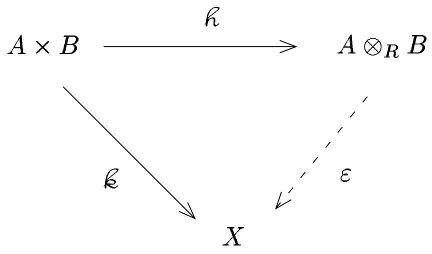
Then $X \cong A \otimes_R B$.

Proof

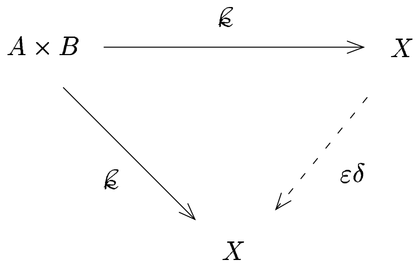
By the assumed property for X, there exists a unique homomorphism $\delta : X \to A \otimes_R B$ so that the diagram commutes:



By the assumed property for $A \otimes_R B$, there exists a unique homomorphism $\varepsilon : A \otimes_R B \to X$ so that the diagram commutes:



Then we have the commutative diagram:



Since $(\varepsilon\delta)\mathscr{k} = \mathscr{k}$ and $\mathit{id}_X \circ \mathscr{k} = \mathscr{k}$, by uniqueness of the homomorphism in the Universal Property, $\varepsilon\delta = \mathit{id}_X$.

Similarly, $\delta\varepsilon = \mathit{id}_{A\otimes_R B}$.

Thus δ is an isomorphism, and $X \cong A \otimes_R B$ via δ.