THE MANIFOLD OF ALL LINES IN $\mathbb{R}^3$

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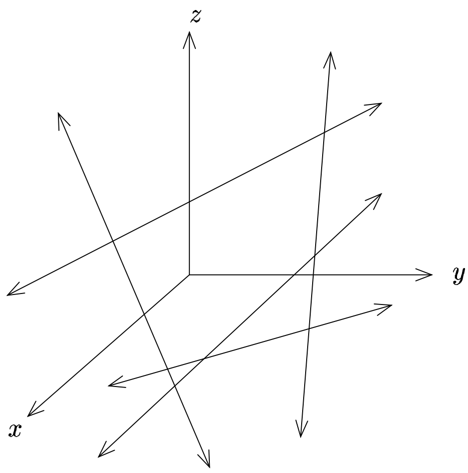
This was discussed by Dr. John McCarthy during MTH869 in Spring 2004.

STEP 1 (Set step): Let $X =$the set of all lines in $\mathbb{R}^3$.



STEP 2 (Topological space step):

Note: If $l \in X$, then $l \notin \mathbb{R}^3$, so we **can't** give X the subspace topology from the standard topology on $\mathbb{R}^3$.

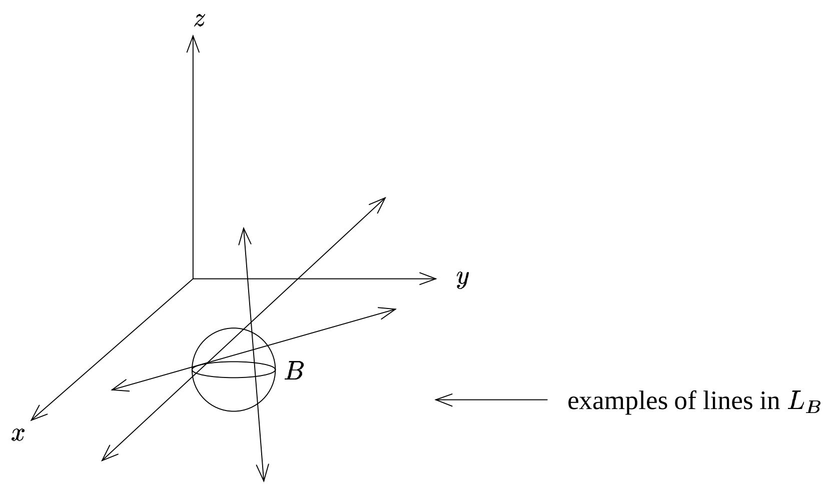
Thus we need to address the question: How can we topologize X? (What constitutes an open set in X?)

Here is the solution:

Let $\mathcal{B} = \{B(x, \varepsilon) : \ x = (x_1, x_2, x_3)$ and $x_1, x_2, x_3, \varepsilon \in \mathbb{Q}\}$.

Then $\mathcal{B}$ is a countable basis for $\mathcal{T}_{\mathbb{R}^3}$, the standard topology on $\mathbb{R}^3$.

For each $B \in \mathcal{B}$, define $L_B = \{l \in X : \, l \cap B \neq \emptyset\}$.



Let $\mathcal{L} = \{L_B\}_{B \in \mathcal{B}}$.

Notice that, by construction, $\mathcal{L}$ is a cover of X.

Since $\mathcal{L}$ is a collection of subsets of X, $\mathcal{L}$ is a **subbasis** for a topology $\mathcal{T}$ on X.

Let $\mathcal{T}$ be the unique topology generated by the subbasis $\mathcal{L}$.

This makes $(X, \mathcal{T})$ into a **topological space**.

STEP 3 (Topological manifold step):

Claim 1: $(X, \mathcal{T})$ is **second countable**

Proof

Since $\mathcal{B}$ is countable, $\mathcal{L}$ is countable.

Since $\mathcal{T}$ has a countable subbasis, $\mathcal{T}$ has a countable basis. This proves the claim.

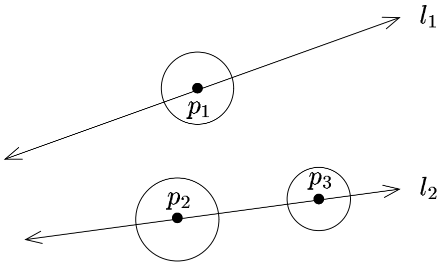
Claim 2: $(X, \mathcal{T})$ is **Hausdorff**

Proof

Let $l_1, l_2 \in X$ with $l_1 \neq l_2$.

Let $p_1 \in l_1$, $p_2 \in l_2$, $p_3 \in l_2$.

To construct open sets in X that separate l_1 and l_2, we consider open balls around p_1, p_2, and p_3 "small enough" such that all lines passing through "the p_1 ball" are separated away from the lines that pass through "the p_2 and p_3 balls". This is the concept we use to construct separating open sets in X.



Since we don't want any lines to intersect all 3 balls, we consider a function that determines when three points lie on the same line.

Thus, let $f : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(x,y,z) = [d(x,y) + d(y,z) - d(x,z)] \cdot [d(y,x) + d(x,z) - d(y,z)] \cdot [d(y,z) + d(z,x) - d(y,x)]$$

Note: $f(x,y,z) = 0$ iff x, y, z all lie on the same line.

Since $l_1 \neq l_2$, we have that p_1, p_2, and p_3 do not lie on the same line.

Thus $f(p_1, p_2, p_3) \neq 0$.

Since the metric d is continuous, f is a continuous function.

Thus there exists an open neighborhood U of (p_1, p_2, p_3) so that
$f(x_1, x_2, x_3) \neq 0$, for all $(x_1, x_2, x_3) \in U$.

Now $\{U_1 \times U_2 \times U_3 : U_i^{open} \subseteq \mathbb{R}^3\}$ is a basis for the standard product topology on $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, so there exists $W_i^{open} \subseteq \mathbb{R}^3$ for $i = 1, 2, 3$ such that $(p_1, p_2, p_3) \in W_1 \times W_2 \times W_3 \subseteq U$.

Thus $p_1 \in W_1$, $p_2 \in W_2$, and $p_3 \in W_3$.

Since $\mathcal{B}$ is a basis for $\mathcal{T}_{\mathbb{R}^3}$, there exists $y_1, y_2, y_3 \in \mathbb{Q}^3$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{Q}$ such that
$p_1 \in B(y_1, \varepsilon_1) \subseteq W_1$, $p_2 \in B(y_2, \varepsilon_2) \subseteq W_2$, and $p_3 \in B(y_3, \varepsilon_3)$.

Thus $l_1 \cap B(y_1, \varepsilon_1) \neq \emptyset$, $l_2 \cap B(y_2, \varepsilon_2) \neq \emptyset$, and $l_2 \cap B(y_3, \varepsilon_3) \neq \emptyset$.

Then we have $l_1 \in L_{B(y_1,\varepsilon_1)}$, $l_2 \in L_{B(y_2,\varepsilon_2)}$, and $l_2 \in L_{B(y_3,\varepsilon_3)}$.

Let $U_1 = L_{B(y_1,\varepsilon_1)}$ and $U_2 = L_{B(y_2,\varepsilon_2)} \cap L_{B(y_3,\varepsilon_3)}$.

Then $l_1 \in U_1 \in \mathcal{T}$ and $l_2 \in U_2 \in \mathcal{T}$.

Since U_1 is open in X containing l_1 and U_2 is open in X containing l_2,
it remains to show that $U_1 \cap U_2 = \emptyset$.

Now assume, by way of contradiction, that $U_1 \cap U_2 \neq \emptyset$.

Then let $l \in U_1 \cap U_2$. Then $l \in L_{B(y_1,\varepsilon_1)} \cap L_{B(y_2,\varepsilon_2)} \cap L_{B(y_3,\varepsilon_3)}$.

Hence $l \cap B(y_1, \varepsilon_1) \neq \emptyset$, $l \cap B(y_2, \varepsilon_2) \neq \emptyset$, and $l \cap B(y_3, \varepsilon_3) \neq \emptyset$.

Thus there exists $q_1, q_2, q_3 \in l$ such that $q_1 \in B(y_1, \varepsilon_1)$, $q_2 \in B(y_2, \varepsilon_2)$ and $q_3 \in B(y_3, \varepsilon_3)$.

Then $(q_1, q_2, q_3) \in B(y_1, \varepsilon_1) \times B(y_2, \varepsilon_2) \times B(y_3, \varepsilon_3) \subseteq W_1 \times W_2 \times W_3 \subseteq U$.

Since $(q_1, q_2, q_3) \in U$, $\mathscr{f}(q_1, q_2, q_3) \neq 0$.

Hence q_1, q_2, and q_3 do not lie on the same line, a contradiction.

Thus $U_1 \cap U_2 = \emptyset$, so $(X, \mathcal{T})$ is Hausdorff. This proves the claim.

Claim 3: $(X, \mathcal{T})$ is **locally Euclidean**

Proof

Let $l \in X$.

Since l is a line, there exists $(x_1, y_1, z_1), (x_2, y_2, z_2) \in l$ such that $(x_1, y_1, z_1) \neq (x_2, y_2, z_2)$.

Let $U_1 = \{l \in X : \exists (x_1, y_1, z_1), (x_2, y_2, z_2) \in l$ such that $x_1 \neq x_2\}$,

$U_2 = \{l \in X : \exists (x_1, y_1, z_1), (x_2, y_2, z_2) \in l$ such that $y_1 \neq y_2\}$, and

$U_3 = \{l \in X : \exists (x_1, y_1, z_1), (x_2, y_2, z_2) \in l$ such that $z_1 \neq z_2\}$.

Then $l \in U_1$, $l \in U_2$, or $l \in U_3$.

Subclaim 1: For all $i \in \{1, 2, 3\}$, U_i is open in X

Proof

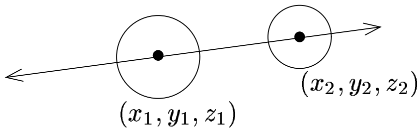
We will prove that U_1 is open in X.

The proofs that U_2 and U_3 are open in X are similar.

Let $l_1 \in U_1$.

Then there exists $a = (x_1, y_1, z_1) \in l$ and $b = (x_2, y_2, z_2) \in l$ such that $x_1 \neq x_2$.

To construct our open neighborhood of l_1 in X, we consider open balls around a and b. We choose these "small enough" such that all lines passing through these balls form an open set in X that is contained in U_1.



Let $\varepsilon = \dfrac{|x_2 - x_1|}{3}$.

Notice that $a \in B(a, \varepsilon)$ and $b \in B(b, \varepsilon)$. Since $B(a, \varepsilon)$ and $B(b, \varepsilon)$ are open in $\mathbb{R}^3$ and $\mathcal{B}$ is a basis for $\mathcal{R}^3$, there exist $c_1, \varepsilon_1, c_2, \varepsilon_2 \in \mathbb{Q}$ such that $a \in B(c_1, \varepsilon_1) \subseteq B(a, \varepsilon)$ and $b \in B(c_2, \varepsilon_2) \subseteq B(b, \varepsilon)$.

Then $l \cap B(c_1, \varepsilon_1) \neq \emptyset$ and $l \cap B(c_2, \varepsilon_2) \neq \emptyset$.

Hence $l \in L_{B(c_1, \varepsilon_1)}$ and $l \in L_{B(c_2, \varepsilon_2)}$.

Thus $l \in L_{B(c_1, \varepsilon_1)} \cap L_{B(c_2, \varepsilon_2)}$.

Since $L_{B(c_1, \varepsilon_1)} \cap L_{B(c_2, \varepsilon_2)} \in \mathcal{T}$, i.e. open in X, we have produced an open neighborhood of l. We will have succeeded in showing that U_1 is open, if in fact $L_{B(c_1, \varepsilon_1)} \cap L_{B(c_2, \varepsilon_2)} \subseteq U_1$.

Thus it remains to show that $L_{B(c_1, \varepsilon_1)} \cap L_{B(c_2, \varepsilon_2)} \subseteq U_1$.

Let $m \in L_{B(c_1, \varepsilon_1)} \cap L_{B(c_2, \varepsilon_2)}$.

Then $m \cap B(c_1, \varepsilon_1) \neq \emptyset$ and $m \cap B(c_2, \varepsilon_2) \neq \emptyset$.

Let $(r_1, s_1, t_1) \in m \cap B(c_1, \varepsilon_1)$ and $(r_2, s_2, t_2) \in m \cap B(c_2, \varepsilon_2)$.

Then $(r_1, s_1, t_1) \in B(c_1, \varepsilon_1) \subseteq B(a, \varepsilon)$ and $(r_2, s_2, t_2) \in B(c_2, \varepsilon_2) \subseteq B(b, \varepsilon)$.

Then we have

$$\begin{aligned} 3\varepsilon = |x_2 - x_1| &\leq |x_2 - r_2| + |r_2 - r_1| + |r_1 - x_1| \\ &\leq d((r_2, s_2, t_2), b) + |r_2 - r_1| + d((r_1, s_1, t_1), a) \\ &< \varepsilon + |r_2 - r_1| + \varepsilon \end{aligned}$$

Thus $|r_2 - r_1| > \varepsilon$, so $r_1 \neq r_2$.

Hence we have $(r_1, s_1, t_1), (r_2, s_2, t_2) \in m$ such that $r_1 \neq r_2$.

Thus $m \in U_1$.

Thus $L_{B(c_1,\varepsilon_1)} \cap L_{B(c_2,\varepsilon_2)} \subseteq U_1$ as desired.

This proves subclaim 1.

Subclaim 2:

$U_1 = \{l \in X : \ l \cap (\{0\} \times \mathbb{R} \times \mathbb{R}) \neq \emptyset$ and $l \cap (\{1\} \times \mathbb{R} \times \mathbb{R}) \neq \emptyset\}$,

$U_2 = \{l \in X : \ l \cap (\mathbb{R} \times \{0\} \times \mathbb{R}) \neq \emptyset$ and $l \cap (\mathbb{R} \times \{1\} \times \mathbb{R}) \neq \emptyset\}$, and

$U_3 = \{l \in X : \ l \cap (\mathbb{R} \times \mathbb{R} \times \{0\}) \neq \emptyset$ and $l \cap (\mathbb{R} \times \mathbb{R} \times \{1\}) \neq \emptyset\}$

Proof

For a given line in U_i, the existence of two points with differing ith coordinates gives rise to a parametric equation for the line which can be used to guarantee the intersection with the planes indicated above, and similarly for the converse direction. The exact details of the proof are left as a simple exercise.

Let $V_1 = V_2 = V_3 = \mathbb{R}^4$.

Now define maps $\varphi_i : \ U_i \to V_i$ as follows:

$\varphi_1 : \ U_1 \to V_1$:

Let $l \in U_1$. Then $l \cap (\{0\} \times \mathbb{R} \times \mathbb{R}) = (0, v_1, w_1)$ and $l \cap (\{1\} \times \mathbb{R} \times \mathbb{R}) = (1, v_2, w_2)$ for some $v_1, w_1, v_2, w_2 \in \mathbb{R}$. Then let $\varphi_1(l) = (v_1, w_1, v_2, w_2)$.

$\varphi_2 : U_2 \to V_2$:

Let $l \in U_2$. Then $l \cap (\mathbb{R} \times \{0\} \times \mathbb{R}) = (u_1, 0, w_1)$ and $l \cap (\mathbb{R} \times \{1\} \times \mathbb{R}) = (u_2, 1, w_2)$ for some $u_1, w_1, u_2, w_2 \in \mathbb{R}$. Then let $\varphi_2(l) = (u_1, w_1, u_2, w_2)$.

$\varphi_3 : U_3 \to V_3$:

Let $l \in U_3$. Then $l \cap (\mathbb{R} \times \mathbb{R} \times \{0\}) = (u_1, v_1, 0)$ and $l \cap (\mathbb{R} \times \mathbb{R} \times \{1\}) = (u_2, v_2, 1)$ for some $u_1, v_1, u_2, v_2 \in \mathbb{R}$. Then let $\varphi_3(l) = (u_1, v_1, u_2, v_2)$.

To complete the proof that $(X, \mathcal{T})$ is locally Euclidean, it suffices to show that φ_i is a homeomorphism for all $i \in \{1, 2, 3\}$.

It is clear from the definition that φ_i, for $i \in \{1, 2, 3\}$, is a bijection.

Thus, we need to show that, for $i \in \{1, 2, 3\}$, φ_i and ${\varphi_i}^{-1}$ are continuous.

Subclaim 3: For all $i \in \{1, 2, 3\}$, φ_i is continuous

Proof

We will prove that φ_1 is continuous. The proofs that φ_2 and φ_3 are continuous are similar.

This proof will be similar in spirit, although slightly more complicated, to the proof showing that U_1 was open in X.

Now $\varphi_1 : U_1 \to V_1$, and let U be open in $V_1 = \mathbb{R}^4$.

We want to show that ${\varphi_1}^{-1}(U)$ is open in U_1.

Let $l \in {\varphi_1}^{-1}(U)$

Since ${\varphi_1}^{-1}(U) \subseteq U_1$, there exists $a = (0, y_1, z_1) \in l$ and $b = (1, y_2, z_2) \in l$ for some $y_1, z_1, y_2, z_2 \in l$.

Then $\varphi_1(l) = (y_1, z_1, y_2, z_2) \in U$.

Since U is open in $\mathbb{R}^4$, there exists $\varepsilon' > 0$ such that $B(\varphi_1(l), \varepsilon') \subseteq U$.

Let $M = max\{1, |y_2 - y_1|, |z_2 - z_1|\}$ and let $\varepsilon = min\{\frac{1}{3}, \frac{\varepsilon'}{23M}\}$.

Now $a \in B(a, \varepsilon)$ and $b \in B(b, \varepsilon)$, so since $\mathcal{B}$ is a basis for $\mathbb{R}^3$, there exists $c_1, \varepsilon_1, c_2, \varepsilon_2 \in \mathbb{Q}$ such that $a \in B(c_1, \varepsilon_1) \subseteq B(a, \varepsilon)$ and $b \in B(c_2, \varepsilon_2) \subseteq B(b, \varepsilon)$.

Hence $l \in L_{B(c_1,\varepsilon_1)}$ and $l \in L_{B(c_2,\varepsilon_2)}$.

Thus $l \in L_{B(c_1,\varepsilon_1)} \cap L_{B(c_2,\varepsilon_2)}$.

We now show that $L_{B(c_1,\varepsilon_1)} \cap L_{B(c_2,\varepsilon_2)} \subseteq U_1$ and $L_{B(c_1,\varepsilon_1)} \cap L_{B(c_2,\varepsilon_2)} \subseteq {\varphi_1}^{-1}(U)$.

Let $m \in L_{B(c_1,\varepsilon_1)} \cap L_{B(c_2,\varepsilon_2)}$.

Then $m \cap B(c_1,\varepsilon_1) \neq \emptyset$ and $m \cap B(c_2,\varepsilon_2) \neq \emptyset$.

Thus there exists $(r_1, s_1, t_1) \in m \cap B(c_1,\varepsilon_1)$ and $(r_2, s_2, t_2) \in m \cap B(c_2,\varepsilon_2)$.

Then $(r_1, s_1, t_1) \in B(c_1,\varepsilon_1) \subseteq B(a,\varepsilon)$ and $(r_2, s_2, t_2) \in B(c_2,\varepsilon_2) \subseteq B(b,\varepsilon)$.

Hence

$$\begin{aligned}
1 &\leq |1 - r_2| + |r_2 - r_1| + |r_1 - 0| \\
&\leq d((r_2, s_2, t_2), b) + |r_2 - r_1| + d((r_1, s_1, t_1), a) \\
&\leq \varepsilon + |r_2 - r_1| + \varepsilon \\
&\leq \frac{1}{3} + |r_2 - r_1| + \frac{1}{3}
\end{aligned}$$

Thus $|r_2 - r_1| \geq \frac{1}{3}$, so $r_1 \neq r_2$. Thus $m \in U_1$.

Hence $L_{B(c_1,\varepsilon_1)} \cap L_{B(c_2,\varepsilon_2)} \subseteq U_1$.

Letting $m \in L_{B(c_1,\varepsilon_1)} \cap L_{B(c_2,\varepsilon_2)}$, we get the same information as above.

Furthermore, since $m \in U_1$, there exists $v_1, w_1 \in \mathbb{R}$ such that for some $t' \in \mathbb{R}$, $(0, v_1, w_1) = (r_1, s_1, t_1) + t'(r_2 - r_1, s_2 - s_1, t_2 - t_1)$.

Then $0 = r_1 + t'(r_2 - r_1)$. Since $r_1 \neq r_2$, $t' = \frac{r_1}{r_1 - r_2}$.

Now $v_1 = s_1 + t'(s_2 - s_1) = s_1 + \frac{r_1}{r_1 - r_2}(s_2 - s_1)$
and $w_1 = t_1 + t'(t_2 - t_1) = t_1 + \frac{r_1}{r_1 - r_2}(t_2 - t_1)$.

Similarly, since $m \in U_1$, there exists $v_2, w_2 \in \mathbb{R}$ such that for some $t'' \in \mathbb{R}$, $(1, v_2, w_2) = (r_2, s_2, t_2) + t''(r_1 - r_2, s_1 - s_2, t_1 - t_2)$.

Then, similarly, $v_2 = s_2 + \frac{1 - r_2}{r_1 - r_2}(s_1 - s_2)$ and $w_2 = t_2 + \frac{1 - r_2}{r_1 - r_2}(t_1 - t_2)$.

Now $d(a, (r_1, s_1, t_1)) = d((0, y_1, z_1), (r_1, s_1, t_1)) < \varepsilon$,
so $|s_1 - y_1| < \varepsilon$, and $|t_1 - z_1| < \varepsilon$.

Also $d(b,(r_2,s_2,t_2)) = d((1,y_2,z_2),(r_2,s_2,t_2)) < \varepsilon$,
so $|s_2 - y_2| < \varepsilon$, and $|t_2 - z_2| < \varepsilon$.

Then $|s_1 - s_2| \leq |s_1 - y_1| + |y_1 - y_2| + |y_2 - s_2| < \varepsilon + M + \varepsilon = 2\varepsilon + M$.

Similarly, $|t_2 - t_1| < 2\varepsilon + M$.

Then

$$\begin{aligned}
d(a,(r_1,s_1,t_1)) &= d((0,v_1,w_1),(r_1,s_1,t_1)) \\
&= \sqrt{r_1^2 + (s_1 - v_1)^2 + (t_1 - w_1)^2} \\
&= \sqrt{r_1^2 + \left(\frac{r_1}{r_1 - r_2}\right)^2 (s_1 - s_2)^2 + \left(\frac{r_1}{r_1 - r_2}\right)^2 (t_1 - t_2)^2} \\
&< \sqrt{\varepsilon^2 + \frac{\varepsilon^2}{(\frac{1}{3})^2}(2\varepsilon + M)^2 + \frac{\varepsilon^2}{(\frac{1}{3})^2}(2\varepsilon + M)^2} \\
&= \sqrt{\varepsilon^2 + 18\varepsilon^2(2\varepsilon + M)^2} \\
&< \sqrt{\varepsilon^2(2\varepsilon + M)^2 + 18\varepsilon^2(2\varepsilon + M)^2} \\
&= \sqrt{19\varepsilon^2(2\varepsilon + M)^2} \\
&< \sqrt{25\varepsilon^2(2\varepsilon + M)^2} \\
&= 5\varepsilon(2\varepsilon + M) \\
&\leq 5\varepsilon\left(\frac{2}{3} + M\right) \\
&\leq 5\varepsilon(M + M) \\
&= 10M\varepsilon
\end{aligned}$$

Then

$$\begin{aligned}
d((0,y_1,z_1),(0,v_1,w_1)) &\leq d((0,y_1,z_1),(r_1,s_1,t_1)) + d((r_1,s_1,t_1),(0,v_1,w_1)) \\
&< \varepsilon + 10M\varepsilon \\
&\leq M\varepsilon + 10M\varepsilon \\
&= 11M\epsilon
\end{aligned}$$

Similarly, we have that $d((1,y_2,z_2),(1,v_2,w_2)) < 11M\varepsilon$.

Hence we have that
$|y_1 - v_1| < 11M\varepsilon$, $|z_1 - w_1| < 11M\varepsilon$, $|y_2 - v_2| < 11M\varepsilon$, $|z_2 - w_2| < 11M\varepsilon$.

Then

$$\begin{aligned} d((y_1,z_1,y_2,z_2),(v_1,w_1,v_2,w_2)) &= \sqrt{(y_1-v_1)^2+(z_1-w_1)^2+(y_2-v_2)^2+(z_2-w_2)^2} \\ &< \sqrt{(11M\varepsilon)^2+(11M\varepsilon)^2+(11M\varepsilon)^2+(11M\varepsilon)^2} \\ &= \sqrt{4(11M\varepsilon)^2} = 22M\varepsilon \leq 22M\frac{\varepsilon'}{23M} = \frac{22\varepsilon'}{23} < \varepsilon' \end{aligned}$$

Thus $\varphi_1(m) = (v_1, w_1, v_2, w_2) \in B(\varphi_1(l), \varepsilon') \subseteq U$.

Thus $m \in {\varphi_1}^{-1}(U)$.

Hence we have shown that $L_{B(c_1,\varepsilon_1)} \cap L_{B(c_2,\varepsilon_2)} \subseteq {\varphi_1}^{-1}(U)$.

Since $L_{B(c_1,\varepsilon_1)} \cap L_{B(c_2,\varepsilon_2)} \subseteq U_1$, we have that
$L_{B(c_1,\varepsilon_1)} \cap L_{B(c_2,\varepsilon_2)} \cap U_1 = L_{B(c_1,\varepsilon_1)} \cap L_{B(c_2,\varepsilon_2)}$.

Thus we have $l \in L_{B(c_1,\varepsilon_1)} \cap L_{B(c_2,\varepsilon_2)} \cap U_1 \subseteq {\varphi_1}^{-1}(U)$.

Let $W = L_{B(c_1,\varepsilon_1)} \cap L_{B(c_2,\varepsilon_2)} \cap U_1$. Then W is open in U_1.

Thus $l \in W \subseteq {\varphi_1}^{-1}(U)$.

Hence we have shown that ${\varphi_1}^{-1}(U)$ is open in U_1.

Thus φ_1 is continuous, and this proves the subclaim.

Subclaim 4: For all $i \in \{1, 2, 3\}$, ${\varphi_i}^{-1}$ is continuous

Proof

We will prove that ${\varphi_1}^{-1}$ is continuous. The proofs that ${\varphi_2}^{-1}$ and ${\varphi_3}^{-1}$ are continuous are similar.

Now $\varphi_1 : U_1 \to V_1$ so ${\varphi_1}^{-1} : V_1 \to U_1$.

Let U be open in U_1.

Then $U = U_1 \cap V$ for some V that is open in X.

Since U_1 is open in X, we have U open in X.

Then, by definition of $\mathcal{T}$, $U = \bigcup_{\alpha\in\Lambda} \bigcap_{i_\alpha=1}^{n_\alpha} L_{B_{\alpha i_\alpha}}$ for some $L_{B_{\alpha i_\alpha}} \in \mathcal{L}$.

Then,

$$\begin{aligned}
({\varphi_1}^{-1})^{-1}(U) &= ({\varphi_1}^{-1})^{-1}\left(\bigcup_{\alpha\in\Lambda}\bigcap_{i_\alpha=1}^{n_\alpha} L_{B_{\alpha i_\alpha}}\right)\\
&= \bigcup_{\alpha\in\Lambda}\bigcap_{i_\alpha=1}^{n_\alpha} ({\varphi_1}^{-1})^{-1}(L_{B_{\alpha i_\alpha}})\\
&= \bigcup_{\alpha\in\Lambda}\bigcap_{i_\alpha=1}^{n_\alpha} \varphi_1(L_{B_{\alpha i_\alpha}})
\end{aligned}$$

Thus to show that ${\varphi_1}^{-1}$ is continuous, it suffices to show that $\varphi_1(L_{B_{\alpha i_\alpha}})$ is open in $\mathbb{R}^4$, for all α and i_α.

Let $\alpha \in \Lambda$, $n_\alpha \in \mathbb{N}$, and $i_\alpha \in \{1, \cdots, n_\alpha\}$.

Let $x = (u_1, v_1, u_2, v_2) \in \varphi_1(L_{B_{\alpha i_\alpha}})$.

Then there exists $l \in L_{B_{\alpha i_\alpha}}$ such that $\varphi_1(l) = (u_1, v_1, u_2, v_2)$.

Thus $l \cap B_{\alpha i_\alpha} \neq \emptyset$ and $(0, u_1, v_1), (1, u_2, v_2) \in l$.

Then there exists $p = (r, s, t) \in l \cap B_{\alpha i_\alpha} \neq \emptyset$

Then there exists $u \in \mathbb{R}$ such that $p = (r, s, t) = (1-u)(0, u_1, v_1) + u(1, u_2, v_2) \in B_{\alpha i_\alpha}$.

By definition of $L_{B_{\alpha i_\alpha}}$, there exists $q, \varepsilon \in \mathbb{Q}$ such that $B_{\alpha i_\alpha} = B(q, \varepsilon)$.

Then $p \in B(q, \varepsilon)$.

Since $B(q, \varepsilon)$ is open, there exists $\varepsilon' > 0$ such that $B(p, \varepsilon') \subseteq B(q, \varepsilon)$.

Let $M = max\{|u|, |1-u|\}$ and let $\delta = \frac{\varepsilon'}{4M}$.

We will now show that $B(x, \delta) \subseteq \varphi_1(L_{B_{\alpha i_\alpha}})$:

Let $y = (u_1', v_1', u_2', v_2') \in B(x, \delta) \subseteq V_1 = \mathbb{R}^4$.

Then there exists $m \in U_1$ such that $y = \varphi_1(m)$ and $(0, u_1', v_1'), (1, u_2', v_2') \in m$.

Let $p' = (1-u)(0, u_1', v_1') + u(1, u_2', v_2')$, and note that $p' \in m$.

Then

$$\begin{aligned}
d(p,p') &= \sqrt{[(1-u)(u_1-u_1')+u(u_2-u_2')]^2+[(1-u)(v_1-v_1')+u(v_2-v_2')]^2} \\
&\leq |(1-u)(u_1-u_1')+u(u_2-u_2')|+|(1-u)(v_1-v_1')+u(v_2-v_2')| \\
&\leq |1-u||u_1-u_1'|+|u||u_2-u_2'|+|1-u||v_1-v_1'|+|u||v_2-v_2'| \\
&\leq M d(x,y)+M d(x,y)+M d(x,y)+M d(x,y) \\
&= 4M d(x,y) < 4M\delta = 4M\frac{\varepsilon'}{4M} = \varepsilon'
\end{aligned}$$

Thus $p' \in B(p,\varepsilon') \subseteq B(q,\varepsilon) = B_{\alpha i_\alpha}$.

Since $p' \in m$ and $p' \in B_{\alpha i \alpha}$, we have that $m \cap B_{\alpha i_\alpha} \neq \emptyset$.

Thus $m \in L_{B_{\alpha i_\alpha}}$.

Then $y = \varphi_1(m) \in \varphi_1(L_{B_{\alpha i_\alpha}})$.

Thus $B(x,\delta) \subseteq \varphi_1(L_{B_{\alpha i_\alpha}})$, so $\varphi_1(L_{B_{\alpha i_\alpha}})$ is open in $\mathbb{R}^4 = V_1$.

Hence ${\varphi_1}^{-1}$ is continuous.

This proves the subclaim.

Thus $(X,\mathcal{T})$ is locally Euclidean, and this proves the claim.

This makes $(X,\mathcal{T})$ into a **topological manifold**.

STEP 4 (Topological atlas step): Let $\mathcal{A} = \{(U_\alpha, V_\alpha, \varphi_\alpha) : \alpha = 1,2,3\}$

Then, by the work from STEP 3, $\mathcal{A}$ is a topological atlas.

STEP 5 (C^∞-atlas step):

Let $\alpha, \beta \in \{1,2,3\}$.

Let $\tau_{\alpha\beta} = \varphi_\alpha|_{U_\alpha \cap U_\beta}^{\varphi_\alpha(U_\alpha\cap U_\beta)} \circ \left(\varphi_\beta|_{U_\alpha \cap U_\beta}^{\varphi_\beta(U_\alpha\cap U_\beta)}\right)^{-1}$

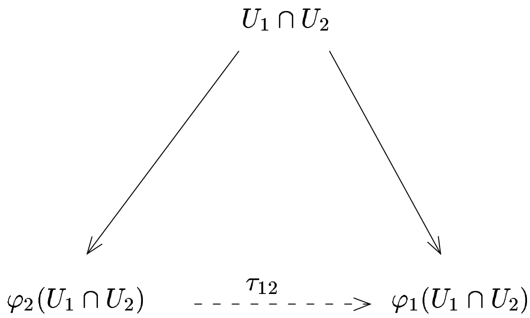
Then $\tau_{\alpha\beta} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$.

We need to show that $\tau_{\alpha\beta}$ is C^∞.

Thus we need to show that τ_{12}, τ_{21}, τ_{13}, τ_{31}, τ_{23}, and τ_{32} are C^∞.

We will show that τ_{12} is C^∞. The other 5 are similar.

Compute τ_{12}:



If $l \in U_1 \cap U_2$, then $l \in U_1$ so there exist $v_1, w_1, v_2, w_2 \in \mathbb{R}$ so that $(0, v_1, w_1) \in l$ and $(1, v_2, w_2) \in l$. Furthermore, $l \in U_2$ so there exist $x_1, z_1, x_2, z_2 \in \mathbb{R}$ so that $(x_1, 0, z_1) \in l$ and $(x_2, 1, z_2) \in l$.

Since $(0, v_1, w_1) \in l$, there exists $s_1 \in \mathbb{R}$ such that $(0, v_1, w_1) = (1 - s_1)(x_1, 0, z_1) + s_1(x_2, 1, z_2)$.

Since $(1, v_2, w_2) \in l$, there exists $s_2 \in \mathbb{R}$ such that $(1, v_2, w_2) = (1 - s_2)(x_1, 0, z_1) + s_2(x_2, 1, z_2)$.

Then, by the first equation, $0 = (1 - s_1)x_1 + s_1 x_2 \;\Rightarrow\; s_1 x_1 - s_1 x_2 = x_1 \;\Rightarrow\; s_1 = \frac{x_1}{x_1 - x_2}$

Note that the last equality above holds, because $l \in U_1$ implies that distinct points have distinct x coordinates.

Then, by the second equation, $1 = (1 - s_2)x_1 + s_2 x_2 \;\Rightarrow\; s_2 x_1 - s_2 x_2 = x_1 - 1 \;\Rightarrow\; s_2 = \frac{x_1 - 1}{x_1 - x_2}$

Then, we have,

$v_1 = s_1 = \frac{x_1}{x_1 - x_2}$

$w_1 = (1 - s_1)z_1 + s_1 z_2 = \frac{-x_2}{x_1 - x_2} \cdot z_1 + \frac{x_1}{x_1 - x_2} \cdot z_2$

$v_2 = s_2 = \frac{x_1 - 1}{x_1 - x_2}$

$w_2 = (1 - s_2)z_1 + s_2 z_2 = \frac{1 - x_2}{x_1 - x_2} \cdot z_1 + \frac{x_1 - 1}{x_1 - x_2} \cdot z_2$.

Then, we have,

$$\tau_{12}(x_1, z_1, x_2, z_2) = \left(\frac{x_1}{x_1 - x_2}, \frac{x_1 z_2 - x_2 z_1}{x_1 - x_2}, \frac{x_1 - 1}{x_1 - x_2}, \frac{(x_1 - 1)z_2 + (1 - x_2)z_1}{x_1 - x_2} \right)$$

Each coordinate of $\tau_{12}(x_1, z_1, x_2, z_2)$ is a rational function in x_1, z_1, x_2, z_2 with nonzero denominator, so τ_{12} is C^∞.

This makes $\mathcal{A}$ into a C^∞**-atlas**.

Step 6 (C^∞-manifold step): Let $M = (X, \mathcal{T}, |\mathcal{A}|)$, where $|\mathcal{A}|$ is the unique maximal atlas containing $\mathcal{A}$.

Then M is a C^∞**-manifold** of dimension 4.

Exercises

1. Let Y =set of all lines in $\mathbb{R}^3$ passing through the origin.
Prove that Y is a smooth submanifold of dimension 2 of M.

2. Let $\Delta = \{(a, b) \in \mathbb{R}^3 \times \mathbb{R}^3 : a = b\}$ (thus Δ is the diagonal in $\mathbb{R}^3 \times \mathbb{R}^3$).
Let $q : (\mathbb{R}^3 \times \mathbb{R}^3)\backslash\Delta \to X$ be defined by $q(a, b)$=the unique line through a and b.
Prove that q is a quotient map.

3. Let P and Q be parallel planes in $\mathbb{R}^3$. Let $U(P, Q) = \{l \in X : l \cap P \neq \emptyset$ and $l \cap Q \neq \emptyset\}$ (thus this is the set of lines hitting both P and Q). Let $\varphi_{(P,Q)} : U(P, Q) \to P \times Q$ be defined by the rule $\varphi_{(P,Q)}(l) = (A, B)$, where $l \cap P = \{A\}$ and $l \cap Q = \{B\}$. Prove that $\varphi_{(P,Q)}$ is a homeomorphism from $U(P, Q)$ to $P \times Q$ (with respect to the subspace topologies inherited from X and $\mathbb{R}^3 \times \mathbb{R}^3$). *Hint: Restrict the quotient map q from Exercise 2 to get a quotient map that looks "similar" to ${\varphi_{(P,Q)}}^{-1}$. This will give continuity of the inverse. For continuity of the function and to show $U(P, Q)$ is open, use the kernel of an appropriate linear transformation.* This homeomorphism gives rise to a more general set of charts than constructed earlier.

4. Show that the topological atlas constructed in Exercise 3 is a C^∞-atlas and gives rise to the same smooth structure for X as constructed earlier.

5. Let $\pi : X \to \mathbb{R}^3$ be defined by $\pi(l)$ =the unique point p on the line l for which
$d((0, 0, 0), p) \leq d((0, 0, 0), q)$, for all q on l (Thus π is the "closest point to the origin" map.)
Prove that π is C^∞ with respect to the smooth structures involved.

6. Let p be a point in $\mathbb{R}^3\backslash\{(0, 0, 0)\}$. Prove that there exists a smooth map $\phi : U \to X$ such that

(i) U is an open neighborhood of p in $\mathbb{R}^3$ and

(ii) $\pi \circ \phi : U \to \mathbb{R}^3$ is equal to the inclusion map $inc : U \to \mathbb{R}^3$

A smooth map $\phi : U \to X$ with the properties described above is called a **local smooth section of the map** $\pi : X \to \mathbb{R}^3$ **near p**. In order to get such a local smooth section, you will need to make sure that your open neighborhood U of p does not contain $(0, 0, 0)$. The reason for this is that it turns out that the map $\pi : X \to \mathbb{R}^3$ does not have any local smooth sections near $(0, 0, 0)$. It only has local smooth sections away from $(0, 0, 0)$.