CONSTRUCTING A C^∞ MANIFOLD

Step 1 (Set step): Start with a set X.

Step 2 (Topological space step): Topologize X by introducing a topology on X.

This makes $(X, \mathcal{T})$ into a **topological space**.

Step 3 (Topological manifold step): Show that

1. $(X, \mathcal{T})$ is **second countable**
2. $(X, \mathcal{T})$ is **Hausdorff**
3. $(X, \mathcal{T})$ is **locally Euclidean**

This makes $(X, \mathcal{T})$ into a **topological manifold**.

Step 4 (Topological atlas step): Let $\mathcal{A} = \{(U_\alpha, V_\alpha, \varphi_\alpha) : \alpha \in \Lambda\}$ where

1. $\forall \alpha \in \Lambda, U_\alpha \in \mathcal{T}$ (i.e. U_α is an open subset of X)
2. $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover of X
3. $\forall \alpha \in \Lambda$, V_α is an open subset of $\mathbb{R}^n$
4. $\forall \alpha \in \Lambda$, $\varphi_\alpha : U_\alpha \to V_\alpha$ is a homeomorphism

Such an $\mathcal{A}$ is called a **topological atlas**, and such an atlas can be constructed since $(X, \mathcal{T})$ is locally Euclidean.

Step 5 (C^∞-atlas step): Show that $\forall \alpha, \beta \in \Lambda$, $(U_\alpha, V_\alpha, \varphi_\alpha) \in \mathcal{A}$ and $(U_\beta, V_\beta, \varphi_\beta) \in \mathcal{A}$ are C^∞**-related**:

Thus let $\tau_{\alpha\beta} = \varphi_\alpha|_{U_\alpha \cap U_\beta}^{\varphi_\alpha(U_\alpha \cap U_\beta)} \circ \left(\varphi_\beta|_{U_\alpha \cap U_\beta}^{\varphi_\beta(U_\alpha \cap U_\beta)}\right)^{-1}$

Then show that $\tau_{\alpha\beta} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is C^∞.

This makes $\mathcal{A}$ into a C^∞**-atlas**.

Step 6 (C^∞-manifold step): Let $M = (X, \mathcal{T}, |\mathcal{A}|)$, where $|\mathcal{A}|$ is the unique maximal atlas containing $\mathcal{A}$.

Then M is a C^∞**-manifold**.