CATEGORIES AND FUNCTORS

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1.1 Categories

A. Motivation

"Common Theme":

> Introduce Groups, Define Group Homomorphisms, Prove Isomorphism Theorems
>
> Introduce Rings, Define Ring Homomorphisms, Prove Isomorphism Theorems
>
> Introduce Module, Define Module Homomorphisms, Prove Isomorphism Theorems

You are just working in a different "category".

In general,

> Introduce "Objects", Define "Morphisms", where morphisms compose like maps.

Intuitively, we think of a "category" as:

1. A collection of "objects"

\+ 2. A collection of "morphisms" (which we think of as maps)*
* Not quite true: See LATER.

\+ 3. A composition procedure for morphisms (which we can think of as function composition)

Now formalize . . .

B. Definition of a Category

A **category** $\mathcal{C}$ is an ordered triple of classes* (X, Y, Z) such that:

1. Elements of X are called **objects**. X is denoted by $X = \mathit{obj}(\mathcal{C})$.

2. Elements of Y are sets. If $M \in Y$, elements of M are called **morphisms**.

   Furthermore the following properties hold:

   A. **There exists a surjective function $\mathit{hom}_{\mathcal{C}} : X \times X \to Y$:**

      Thus for every ordered pair $(A, B) \in X \times X$ of objects, there exists a unique morphism $M \in Y$ and moreover all elements of Y look like $\mathit{hom}_{\mathcal{C}}(A, B)$ for some $A, B \in \mathit{obj}(\mathcal{C})$.

   B. **Elements of Y are disjoint:**

      Thus if $\mathit{hom}_{\mathcal{C}}(A, B)$ and $\mathit{hom}_{\mathcal{C}}(C, D)$ are elements of Y, then

      $$(A, B) \neq (C, D) \implies \mathit{hom}_{\mathcal{C}}(A, B) \cap \mathit{hom}_{\mathcal{C}}(C, D) = \emptyset$$

3. Elements of Z are functions called **compositions**.

   Furthermore the following properties hold:

   A. **If $F \in Z$, F takes the form** $F : \mathit{hom}_{\mathcal{C}}(A, B) \times \mathit{hom}_{\mathcal{C}}(B, C) \to \mathit{hom}_{\mathcal{C}}(A, C)$

      Notation: $F(f, g) = gf$

   B. **Associativity of composition holds:** $h(gf) = (hg)f$

   C. **Identity morphisms exist:**

      For every $A \in \mathit{obj}(\mathcal{C})$, there exists a morphism $1_A \in \mathit{hom}_{\mathcal{C}}(A, A)$ [$\in Y$] such that

      1. $f 1_A = f$ for every $f \in \mathit{hom}_{\mathcal{C}}(A, B)$

      2. $1_A g = g$ for every $g \in \mathit{hom}_{\mathcal{C}}(B, A)$

C. Comments

1. ***Class:** collection

   Set: A is a **set** if A is a class and $\exists$ a class B such that $A \in B$

2. For $A, B \in \mathcal{obj}(\mathcal{C})$ with $A \neq B$, $\mathcal{hom}_{\mathcal{C}}(A, B)$ could be the empty set.

3. For all $A \in \mathcal{obj}(\mathcal{C})$, 1_A is unique:

   Suppose $\overline{1_A} \in \mathcal{hom}_{\mathcal{C}}(A, A)$ with $f\overline{1_A} = f$, $\forall f \in \mathcal{hom}_{\mathcal{C}}(A, B)$ and $\overline{1_A}\, g = g$, $\forall g \in \mathcal{hom}_{\mathcal{C}}(B, A)$

   Then $\overline{1_A} = \overline{1_A}\, 1_A = 1_A$.

4. Morphisms are **not**, in general, maps:

   Let S be a monoid. Let A be a symbol.

   Define $\mathcal{C}_S$: $\mathcal{C}_S = (X, Y, Z)$ where

   $X = \{A\}$

   $Y = \{S\}$ [i.e. $\mathcal{hom}_{\mathcal{C}}(A, A) = S$]

   $Z = \{\cdot\}$ with $\cdot : S \times S \to S$ standard multiplication

   Then morphisms are just elements of S.

5. Despite comment 4, if $f \in \mathcal{hom}_{\mathcal{C}}(A, B)$, we write $f : A \to B$, even though f is not always a function.

D. Examples

1. <u>Set</u>: the category of sets

   objects: sets

   morphisms: functions $f : A \to B$

   (**compositions:** function composition)

2. Grp: the category of groups

   objects: groups

   morphisms: group homomorphisms

3. Ring: the category of rings

4. $_R-$Mod: the category of left R-modules (for fixed R)

5. Mod$-_R$: the category of right R-modules

6. Top: the category of topological spaces

   objects: topological spaces

   morphisms: continuous functions

7. Let Q be a **quasi-ordered** set (i.e. reflexive and transitive relation; quoset + antisymmetry = poset)

   For $x, y \in Q$ with $x \leq y$, define a new symbol i_y^x.

   Define $\mathcal{C}_Q$: $\mathcal{C}_Q = (X, Y, Z)$ where

   $$X = Q$$

   $$Y \implies \text{elements} \quad \mathcal{hom}_{\mathcal{C}}(x, y) \quad \text{with} \quad \mathcal{hom}_{\mathcal{C}}(x, y) = \begin{cases} \{i_y^x\};\ x \leq y \\ \emptyset; \text{ otherwise} \end{cases}$$

   $$Z \implies \text{compositions} \quad i_z^y\, i_y^x = i_z^x \quad \text{whenever } x \leq y \leq z$$

   $\mathcal{C}_Q$ is used in direct and inverse limits.

1.2 Functors

A. Intuition

A functor is thought of as a “function” between categories.

B. Definition of a Covariant Functor

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

A **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a function where:

1. $A \in \mathit{obj}(\mathcal{C}) \implies FA \in \mathit{obj}(\mathcal{D})$
2. $f : A \to B$ morphism in $\mathcal{C}$ $\implies$ $Ff : FA \to FB$ morphism in $\mathcal{D}$
3. $A \xrightarrow{f} B \xrightarrow{g} C$ morphisms in $\mathcal{C}$ $\implies$ $F(gf) = (Fg)(Ff)$
4. For all $A \in \mathit{obj}(\mathcal{C})$, $F(1_A) = 1_{FA}$

C. Definition of a Contravariant Functor

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a function where:

1. $A \in \mathit{obj}(\mathcal{C}) \implies FA \in \mathit{obj}(\mathcal{D})$
2. $f : A \to B$ morphism in $\mathcal{C}$ $\implies$ $Ff : FB \to FA$ morphism in $\mathcal{D}$
3. $A \xrightarrow{f} B \xrightarrow{g} C$ morphisms in $\mathcal{C}$ $\implies$ $F(gf) = (Ff)(Fg)$
4. For all $A \in \mathit{obj}(\mathcal{C})$, $F(1_A) = 1_{FA}$

Note: To define a functor, one must specify what it does to objects **and** what it does to morphisms.

D. Examples of Covariant Functors

1. **Identity Functor:** $F : \mathcal{C} \to \mathcal{C}$ where $FA = A$ and $F\mathscr{f} = \mathscr{f}$

2. **Forgetful Functors:**

   a. F : $\underline{\text{Grp}} \to \underline{\text{Set}}$ where

   $F[(A, \cdot)] = A$ and $\mathscr{f} \in \mathscr{Hom}(A, B) \overset{F}{\mapsto} (\mathscr{f} : A \to B)$

   b. F : $\underline{\text{Ring}} \to \underline{\text{Grp}}$ where

   $F[(A, +, \cdot)] = (A, \cdot)$ and $\mathscr{f} \in \mathscr{Hom}_{\underline{\text{Ring}}}(A, B) \overset{F}{\mapsto} \mathscr{f} \in \mathscr{Hom}_{\underline{\text{Grp}}}(A, B)$

3. **Covariant $\mathscr{Hom}$:** Let $A \in \mathscr{obj}({}_R\!-\!\underline{\text{Mod}})$

   Define $\mathscr{Hom}_R(A, \_\_) : {}_R\!-\!\underline{\text{Mod}} \to \underline{\text{Set}}$ by

   $B \in \mathscr{obj}({}_R\!-\!\underline{\text{Mod}}) \overset{\mathscr{Hom}_R(A,\_\_)}{\longmapsto} \mathscr{Hom}_R(A, B) \in \mathscr{obj}(\underline{\text{Set}})$

   $\mathscr{f} : B \to C \overset{\mathscr{Hom}_R(A,\_\_)}{\longmapsto} \mathscr{Hom}_R(A, \mathscr{f}) : \mathscr{Hom}_R(A, B) \to \mathscr{Hom}_R(A, C)$

   $g \mapsto \mathscr{f}g$

   Aside: For R commutative, $\mathscr{Hom}_R(A, \_\_)$ can take values in ${}_R\!-\!\underline{\text{Mod}}$

4. **Constant Functor:** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $D \in \mathscr{obj}(\mathcal{D})$.

   Define $|\ | : \mathcal{C} \to \mathcal{D}$ by

   $|C| = D$ for every $C \in \mathscr{obj}(\mathcal{C})$

   $|\mathscr{f}| = 1_D$ for every morphism $\mathscr{f}$.

5. **(Algebraic Topology) “Fundamental Group $*_1$ functor”:**

Define $*_1$: Top $\to$ Grp by

$*_1(X) = \pi_1(X)$ (the fundamental group)

$*_1(f : M \to N) = \Big(f_* : \pi_1(X) \to \pi_1(Y)\Big)$ where f_* is the induced homomorphism

E. Examples of Contravariant Functors

1. **Contravariant $\mathcal{Hom}$:** Let $B \in \mathit{obj}({}_R\text{-}\underline{\text{Mod}})$

Define $\mathcal{Hom}_R(\_\_, B) : {}_R\text{-}\underline{\text{Mod}} \to \underline{\text{Set}}$ by

$A \in \mathit{obj}({}_R\text{-}\underline{\text{Mod}}) \overset{\mathcal{Hom}_R(\_\_,B)}{\longmapsto} \mathcal{Hom}_R(A, B) \in \mathit{obj}(\underline{\text{Set}})$

$f : A \to C \overset{\mathcal{Hom}_R(\_\_,B)}{\longmapsto} \mathcal{Hom}_R(f, B) : \mathcal{Hom}_R(C, B) \to \mathcal{Hom}_R(A, B)$, $\quad g \mapsto gf$

Again if R is commutative, $\mathcal{Hom}_R(\_\_, B)$ can take values in ${}_R\text{-}\underline{\text{Mod}}$

2. **(Differential Topology) “k-form pullback $*^k$ functor”:**

Define $*^k$: SmoothMan $\to$ k-forms by

$*^k(M) = \Omega^k(M)$ (smooth k-forms on M)

$*^k(f : M \to N) = \big(f^* : \Omega^k(N) \to \Omega^k(M)\big)$ where f^* is the induced pullback map