7.3A Simplifying Roots and Radicals

A. List of Squares/Cubes/Etc. To Memorize

1. **Squares**

$$\begin{array}{c|c} 2^2 = 4 & 10^2 = 100 \\ 3^2 = 9 & 11^2 = 121 \\ 4^2 = 16 & 12^2 = 144 \\ 5^2 = 25 & 13^2 = 169 \\ 6^2 = 36 & 14^2 = 196 \\ 7^2 = 49 & 15^2 = 225 \\ 8^2 = 64 & 16^2 = 256 \\ 9^2 = 81 & \end{array}$$

2. **Cubes**

$$\begin{array}{c|c} 2^3 = 8 & 5^3 = 125 \\ 3^3 = 27 & 6^3 = 216 \\ 4^3 = 64 & \end{array}$$

3. **Fourth Powers**

$$\begin{array}{c|c} 2^4 = 16 & 4^4 = 256 \\ 3^4 = 81 & 5^4 = 625 \end{array}$$

4. **Fifth Powers**

$$\begin{array}{c|c} 2^5 = 32 & 3^5 = 243 \end{array}$$

5. **Higher Powers**

$$\begin{array}{c|c} 2^6 = 64 & 2^9 = 512 \\ 2^7 = 128 & 2^{10} = 1024 \\ 2^8 = 256 & \end{array}$$

B. Simplifying Roots

To simplify a root, we remove the largest perfect power from inside the root and put it as the base outside.

Note: If you don't remove the **largest**, your root may not be fully simplified!

Example 1: Simplify $\sqrt[3]{320}$.

Solution

We identify the largest number in the cube table that goes into 320.

Since 64 is the largest number to go into 320, we write $\sqrt[3]{320} = \sqrt[3]{64 \cdot 5}$

Since $4^3 = 64$, the 64 comes out as a 4. Thus, we have

Ans $\boxed{4\sqrt[3]{5}}$

Example 2: Simplify $\sqrt[4]{648}$.

Solution

We identify the largest number in the fourth power table that goes into 648.

Since 81 is the largest number to go into 648, we write $\sqrt[4]{648} = \sqrt[4]{81 \cdot 8}$

Since $3^4 = 81$, the 81 comes out as a 3. Thus, we have

Ans $\boxed{3\sqrt[4]{8}}$

Example 3: Simplify $\sqrt{600}$.

Solution

We identify the largest number in the square table that goes into 600.

Since 100 is the largest number to go into 600, we write $\sqrt{600} = \sqrt{100 \cdot 6}$

Since $10^2 = 100$, the 100 comes out as a 10. Thus, we have

Ans $\boxed{10\sqrt{6}}$

C. Simplifying Radicals

1. First simplify the coefficient root.

2. To simplify the variables, use the "divide and remainder" trick: divide the power by the index; the quotient is the power that comes out and the remainder is the power that stays in.

Note:

For simplicity, we will assume during the next few sections that all variables represent positive real numbers so that we don't have absolute value issues. See the comments at the end.

D. Examples

Example 1: Simplify $\sqrt[4]{64x^6y^{13}z^3}$.
Assume that all variables represent positive real numbers.

Solution

1. First simplify the coefficient root:

   The largest fourth power that goes into 64 is $2^4 = 16$.

   Thus, we have $\sqrt[4]{64x^6y^{13}z^3} = \sqrt[4]{16 \cdot 4x^6y^{13}z^3} = 2\sqrt[4]{4x^6y^{13}z^3}$.

2. Now simplify the variables (divide and remainder trick):

   For x, $6 \div 4 = 1\text{R}2$

   For y, $13 \div 4 = 3\text{R}1$

   For z, $3 \div 4 = 0\text{R}3$

   Thus we have $2x^1y^3\sqrt[4]{4x^2y^1z^3}$

Ans $\boxed{2xy^3\sqrt[4]{4x^2yz^3}}$

Example 2: Simplify $\sqrt[3]{625x^5y^7z^{10}w^6}$.
Assume that all variables represent positive real numbers.

Solution

1. First simplify the coefficient root:

   The largest cube that goes into 625 is $5^3 = 125$.

   Thus, we have $\sqrt[3]{625x^5y^7z^{10}w^6} = \sqrt[3]{125 \cdot 5x^5y^7z^{10}w^6} = 5\sqrt[3]{5x^5y^7z^{10}w^6}$.

2. Now simplify the variables (divide and remainder trick):

   For x, $5 \div 3 = 1\text{R}2$

   For y, $7 \div 3 = 2\text{R}1$

   For z, $10 \div 3 = 3\text{R}1$

   For w, $6 \div 3 = 2\text{R}0$

   Thus we have $5x^1y^2z^3w^2\sqrt[3]{5x^2y^1z^1}$

Ans $\boxed{5xy^2z^3w^2\sqrt[3]{5x^2yz}}$

Example 3: Simplify $\sqrt{72x^7y^4z^3}$.
Assume that all variables represent positive real numbers.

Solution

1. First simplify the coefficient root:

   The largest square that goes into 72 is $6^2 = 36$.

Thus, we have $\sqrt{72x^7y^4z^3} = \sqrt{36 \cdot 2x^7y^4z^3} = 6\sqrt{2x^7y^4z^3}$.

2. Now simplify the variables (divide and remainder trick):

For x, $7 \div 2 = 3\text{R}1$

For y, $4 \div 2 = 2\text{R}0$

For z, $3 \div 2 = 1\text{R}1$

Thus we have $6x^3y^2z^1\sqrt{2x^1z^1}$

Ans $\boxed{6x^3y^2z\sqrt{2xz}}$

E. Comments

1. An alternate way to simplify roots is to write the prime factorization of the number, and then do the "divide and remainder" trick like what is used for variables. This procedure is longer, but useful if you can't figure out how to break down the root.

For example, to find $\sqrt[3]{324}$:

$324 = 2 \cdot 162 = 2 \cdot 2 \cdot 81 = 2 \cdot 2 \cdot 9^2 = 2^2 \cdot (3^2)^2 = 2^2 \cdot 3^4$

Thus $\sqrt[3]{324} = \sqrt[3]{2^2 \cdot 3^4}$

Now use the divide and remainder trick:

For 2, we have $2 \div 3 = 0\text{R}2$

For 3, we have $4 \div 3 = 1\text{R}1$

Thus we have $3^1\sqrt[3]{2^2\cdot 3^1} = 3\sqrt[3]{12}$

2. If you use the alternate method to find roots, it doesn't matter "how" you factor the number into prime powers. It is a fact that the final factorization (up to order) must be the same no matter how you do it.

This fact is called the **Fundamental Theorem of Arithmetic**.

3. The justification for the simplification method given in this section comes from the "Radical Product Rule", which will be discussed in section 7.4A.

4. For experts:

If you want to do general simplification, allowing for the possibility that the variables could be negative, then whatever comes out of a radical must be in absolute values whenever the index is **even** (but not when odd). This arises due to the relationship:

$$\boxed{\sqrt[n]{x^n} = \begin{cases} x; \text{ if } n \text{ is odd} \\ |x|; \text{ if } n \text{ is even} \end{cases}}$$

Let's do an example of this general type:

Simplify $\sqrt[4]{32x^{13}y^6z^8}$

1. First simplify the coefficient root:

The largest fourth power that goes into 32 is $2^4 = 16$.

Thus, we have $\sqrt[4]{32x^{13}y^6z^8} = \sqrt[4]{16\cdot 2x^{13}y^6z^8} = 2\sqrt[4]{2x^{13}y^6z^8}$.

2. Now simplify the variables (divide and remainder trick):

   For x, $13 \div 4 = 3\text{R}1$

   For y, $6 \div 4 = 1\text{R}2$

   For z, $8 \div 4 = 2\text{R}0$

   Thus we have $|2x^3y^1z^2|\sqrt[4]{2x^1y^2}$

3. Simplify the absolute value:

   Since 2, x^2, and z^2 are always positive (or zero), we can remove them from the absolute value.

Thus we have $\boxed{2x^2z^2|xy|\sqrt[4]{2xy^2}}$