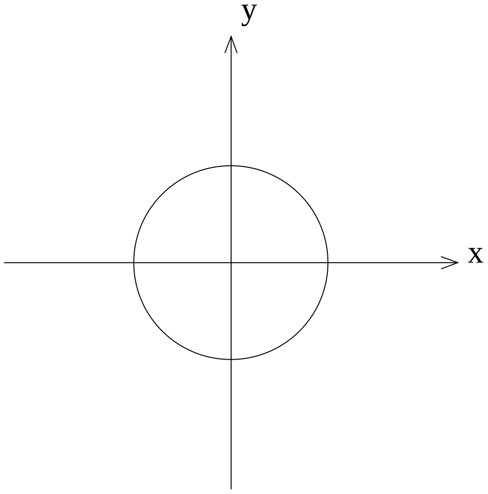
T3.7 Domain and Range of the Trigonometric Functions

A. Sine and Cosine



1. **Domain:**

   Since $w(\theta)$ is defined for any θ with $\cos\theta = x$ and $\sin\theta = y$, there are no domain restrictions.

   Thus $dom(sin) = (-\infty, \infty)$ and $dom(cos) = (-\infty, \infty)$.

2. **Range:**

   The x-coordinate on the circle is smallest at $(-1, 0)$, namely -1; the x-coordinate on the circle is largest at $(1, 0)$, namely 1.

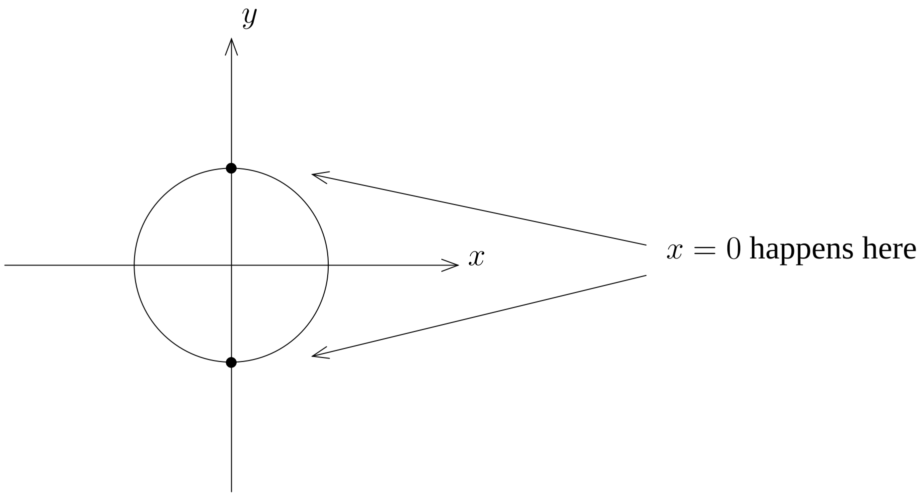
   Hence we can see that $rng(cos) = [-1, 1]$.

   By similar reasoning, we can see that $rng(sin) = [-1, 1]$.

B. Tangent

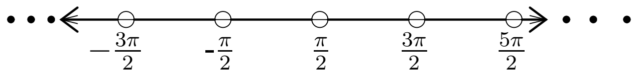
1. **Domain:**

Given $w(\theta) = (x, y)$, we have $\tan\theta = \frac{y}{x}$. Now $\frac{y}{x}$ is undefined when $x = 0$. When does this happen?



Thus $\tan\theta$ is undefined for $\theta = ..., -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...$

What is this in interval notation? To see it, let's plot the allowed values on a number line:



Thus $dom(\tan)$: $... \cup (-\frac{3\pi}{2}, -\frac{\pi}{2}) \cup (-\frac{\pi}{2}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2}) \cup (\frac{3\pi}{2}, \frac{5\pi}{2}) \cup ...$

Note: Each interval has an endpoint being an "odd multiple of $\frac{\pi}{2}$".

Since $2k + 1$ is the formula that generates odd numbers (for k an integer), we recognize that

$dom(\tan)$: union of all intervals of the form $(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2})$, where $k \in \mathbb{Z}$
[k is an integer]

Thus $\mathcal{dom}(\mathcal{tan}) = \bigcup_{k\in\mathbb{Z}} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$.

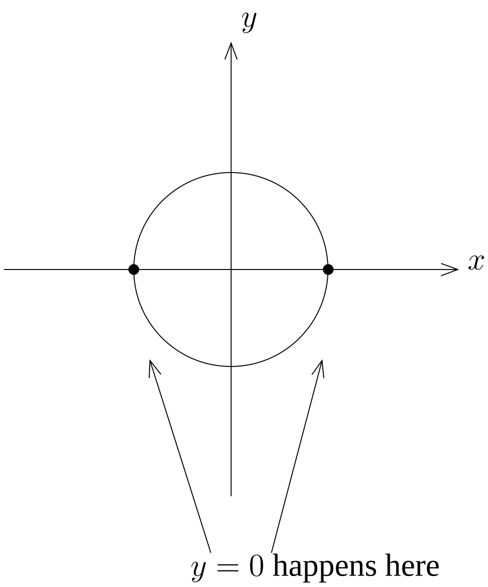
2. **Range:**

Since $\frac{y}{x}$ can be any number, $\mathcal{rng}(\mathcal{tan}) = (-\infty, \infty)$.

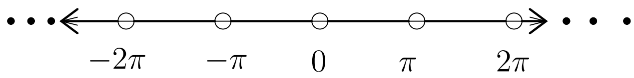
C. Cotangent

1. **Domain:**

This is similar to tangent. Given $\mathcal{w}(\theta) = (x, y)$, we have $\mathcal{cot}\ \theta = \frac{x}{y}$. Now $\frac{x}{y}$ is undefined when $y = 0$. When does this happen?



Thus $\mathcal{cot}\ \theta$ is undefined for $\theta = ..., -2\pi, -\pi, 0, \pi, 2\pi, ...$



Hence $\mathrm{dom}(\mathrm{cot}) = \bigcup_{k\in\mathbb{Z}} (k\pi, (k+1)\pi)$.

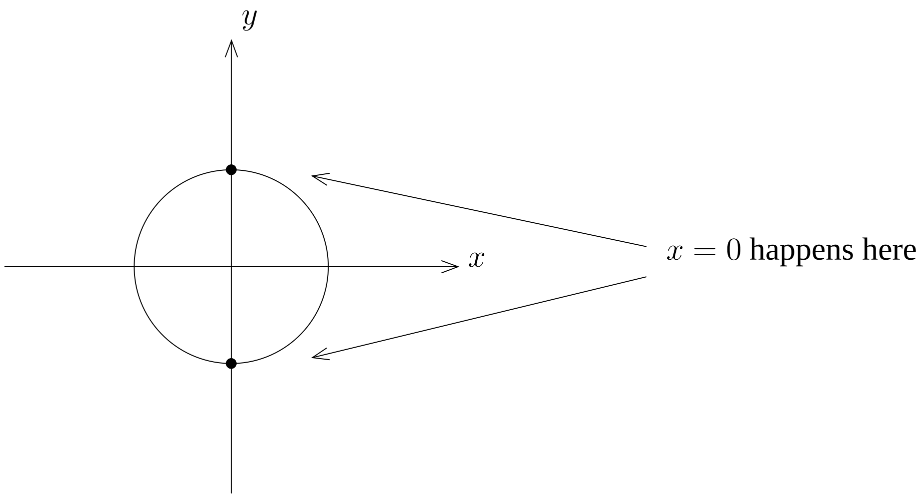
2. **Range:**

Now $\frac{x}{y}$ can be anything, so $\mathrm{rng}(\mathrm{cot}) = (-\infty, \infty)$.

D. Secant

1. **Domain:**

Given $w(\theta) = (x, y)$, we have $\sec\theta = \frac{1}{x}$. Now $\frac{1}{x}$ is undefined when $x = 0$. When does this happen?



So similar to tangent, $\mathrm{dom}(\mathrm{sec}) = \bigcup_{k\in\mathbb{Z}} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$.

2. **Range:**

On the right semicircle, x ranges from 1 down to 0, so $\frac{1}{x}$ ranges from 1 up to ∞.

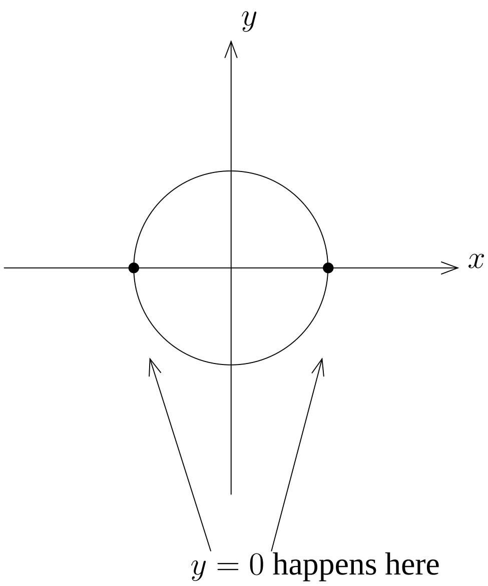
On the left semicircle, x ranges from near 0 to -1, so $\frac{1}{x}$ ranges from $-\infty$ up to -1.

Hence $\mathcal{rng}(\mathcal{sec}) = (-\infty, -1] \cup [1, \infty)$.

E. Cosecant

1. **Domain:**

Given $\mathcal{w}(\theta) = (x, y)$, we have $\mathcal{csc}\,\theta = \frac{1}{y}$. Now $\frac{1}{y}$ is undefined when $y = 0$. When does this happen?



Thus, similar to cotangent, $\mathcal{dom}(\mathcal{cot}) = \bigcup_{k \in \mathbb{Z}} (k\pi, (k+1)\pi)$.

2. **Range:**

By the same reasoning as for secant, we get $\mathcal{rng}(\mathcal{csc}) = (-\infty, -1] \cup [1, \infty)$.

F. Summary

	Domain	Range
sin	$(-\infty, \infty)$	$[-1, 1]$
cos	$(-\infty, \infty)$	$[-1, 1]$
tan	$\bigcup_{k\in\mathbb{Z}} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$	$(-\infty, \infty)$
cot	$\bigcup_{k\in\mathbb{Z}} (k\pi, (k+1)\pi)$	$(-\infty, \infty)$
sec	$\bigcup_{k\in\mathbb{Z}} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$	$(-\infty, -1] \cup [1, \infty)$
csc	$\bigcup_{k\in\mathbb{Z}} (k\pi, (k+1)\pi)$	$(-\infty, -1] \cup [1, \infty)$

Note: To help remember the table, we remember that

1. tan and sec are undefined at odd multiples of $\frac{\pi}{2}$.

2. cot and csc are undefined at multiples of π.