T3.4 The Wrapping Function At Multiples of $\frac{\pi}{4}$

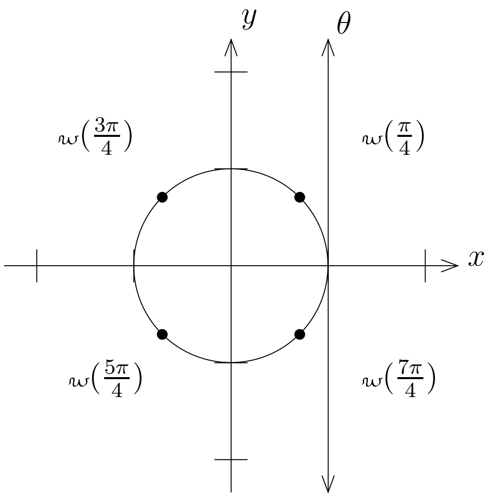
A. Introduction

Evaluating $w(\theta)$ for θ being a multiple of π or $\frac{\pi}{2}$ is direct. However, we need a rule for evaluating $w(\theta)$ when θ is a multiple of $\frac{\pi}{4}$.

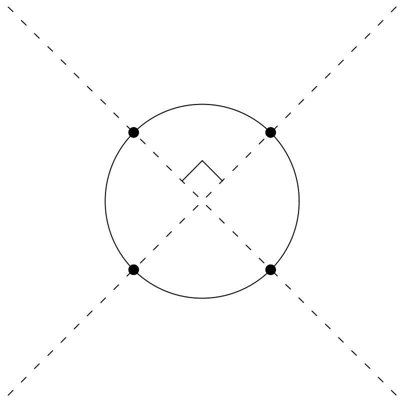
We will derive the $\frac{\pi}{4}$ rule in six easy steps.

B. Derivation of the $\frac{\pi}{4}$ Rule

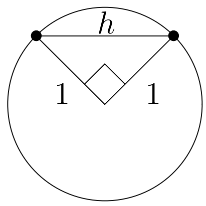
Step 1: Note that if θ is a multiple of $\frac{\pi}{4}$ (lowest terms), then $w(\theta)$ is in one of 4 spots.



Step 2: Drawing diagonal lines through the points yield perpendicular lines, since the points are vertices of a square



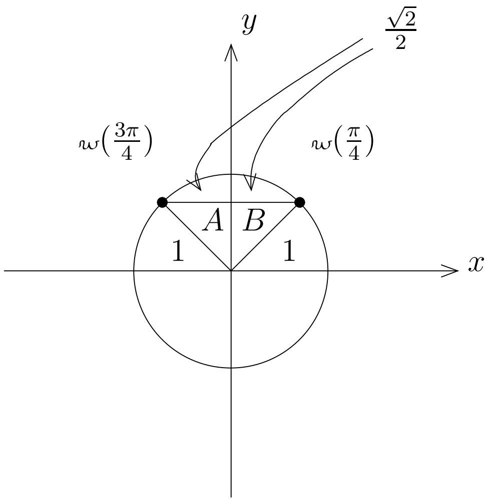
Step 3: Connecting the top two points creates a right triangle with sides having length 1



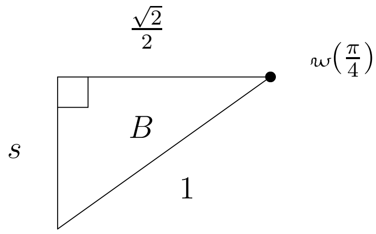
We can use the Pythagorean Theorem to find h:

$$1^2 + 1^2 = h^2 \;\Rightarrow\; h^2 = 2 \;\Rightarrow\; h = \sqrt{2} \text{ (since } h \geq 0\text{)}$$

Step 4: By symmetry, the y axis bisects the triangle into two with top edge length $\frac{\sqrt{2}}{2}$



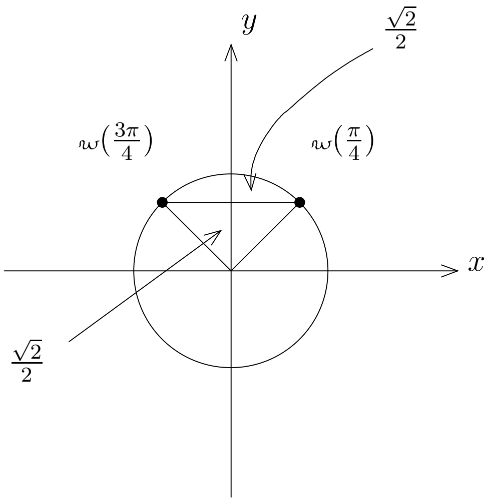
Step 5: We examine triangle B



We can use the Pythagorean Theorem again to find s:

$$s^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1^2 \;\Rightarrow\; s^2 + \frac{2}{4} = 1 \;\Rightarrow\; s^2 = \frac{2}{4} \;\Rightarrow\; s = \frac{\sqrt{2}}{2}$$

Step 6: We now have the following picture, from which we can read off $\omega(\frac{\pi}{4})$



Thus $\boxed{\omega(\frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})}$. By symmetry, we get the $\frac{\pi}{4}$ rule.

C. $\frac{\pi}{4}$ Rule

By symmetry, the x and y coordinates of $\omega(\theta)$ for θ being a multiple of $\frac{\pi}{4}$ are $\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{2}}{2}$ **with appropriate signs**.

D. Strategy

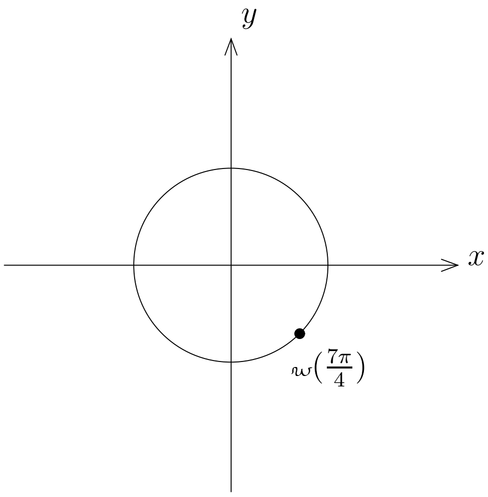
Locate the point on the unit circle, and then use the rule based on the picture.

E. Examples

Example 1: Evaluate $\mathscr{w}(\frac{7\pi}{4})$

Solution

First locate $\mathscr{w}(\frac{7\pi}{4})$ on the unit circle:



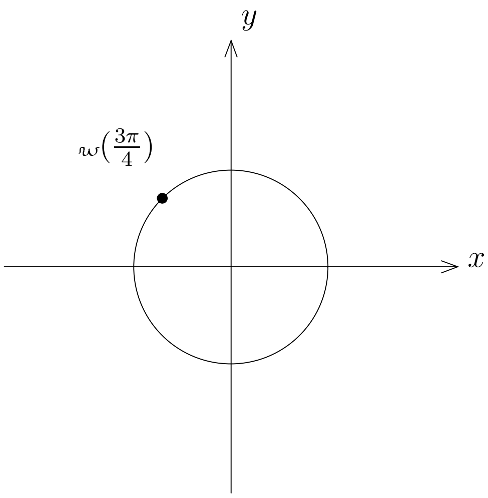
Since we see that to locate the point, we must have positive x and negative y, we have that

Ans $\boxed{\mathscr{w}(\frac{7\pi}{4}) = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})}$

Example 2: Evaluate $w(\frac{3\pi}{4})$

Solution

First locate $w(\frac{3\pi}{4})$ on the unit circle:



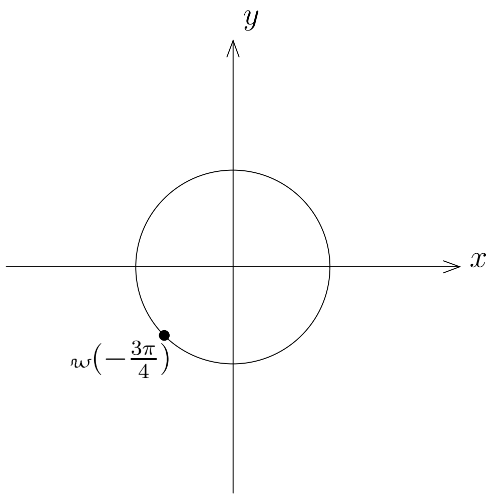
Since we see that to locate the point, we must have negative x and positive y, we have that

Ans $\boxed{w(\frac{3\pi}{4}) = (\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2})}$

Example 3: Evaluate $w(-\frac{3\pi}{4})$

Solution

First locate $w(-\frac{3\pi}{4})$ on the unit circle:



Since we see that to locate the point, we must have negative x and negative y, we have that

Ans $\boxed{w(-\frac{3\pi}{4}) = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})}$