T3.10 Elementary Trigonometric Relationships

A. Discussion

Given $\omega(\theta) = (x, y)$, we have

$$\cos\theta = x \qquad \sec\theta = \frac{1}{x}$$

$$\sin\theta = y \qquad \csc\theta = \frac{1}{y}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

We see that

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

This gives rise to some fundamental identities.

B. Reciprocal Identities

1. $\boxed{\sec\theta = \frac{1}{\cos\theta} \quad \text{and} \quad \cos\theta = \frac{1}{\sec\theta}}$

2. $\boxed{\csc\theta = \frac{1}{\sin\theta} \quad \text{and} \quad \sin\theta = \frac{1}{\csc\theta}}$

3. $\boxed{\cot\theta = \frac{1}{\tan\theta} \quad \text{and} \quad \tan\theta = \frac{1}{\cot\theta}}$

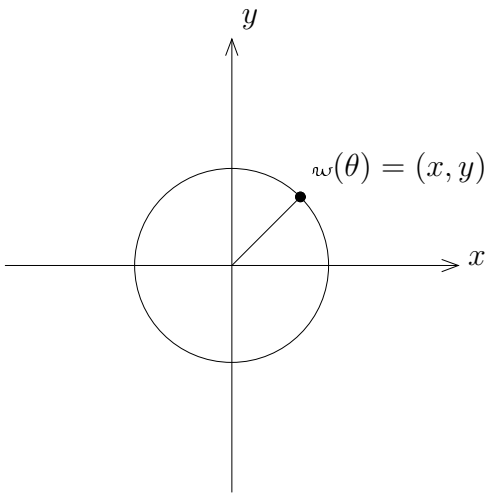
C. Quotient Identities

Using the definitions again, we get

1. $\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$

2. $\boxed{\cot\theta = \frac{\cos\theta}{\sin\theta}}$

D. The Pythagorean Identity



Note: $x^2 + y^2 = 1$ (because we have a unit circle)

Since we have that $\cos\theta = x$ and $\sin\theta = y$, the equation becomes

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

Shorthand: $(\cos\theta)^2 = \cos^2\theta$

Warning: $\cos\theta^2$ **does not** mean $(\cos\theta)^2$; $\cos\theta^2$ means $\cos(\theta^2)$

E. Pythagorean Identities

Thus we have **Pythagorean I:** $\boxed{\cos^2\theta + \sin^2\theta = 1}$

Now, if we divide both sides of Pythagorean I by $\cos^2\theta$, we get:

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

Thus we have **Pythagorean II:** $\boxed{1 + \tan^2\theta = \sec^2\theta}$

Dividing both sides of Pythagorean I by $\sin^2\theta$, we get:

$$\frac{\cos^2\theta}{\sin^2\theta} + 1 = \frac{1}{\sin^2\theta}$$

Thus we have **Pythagorean III:** $\boxed{\cot^2\theta + 1 = \csc^2\theta}$

G. Examples

Example 1: If $\sin\theta = \frac{2}{5}$, what are the possible values of $\cos\theta$?

Solution

We use Pythagorean I: $\cos^2\theta + \sin^2\theta = 1$

Thus,

$$\begin{aligned} \cos^2\theta + \left(\frac{2}{5}\right)^2 &= 1 \\ \cos^2\theta + \frac{4}{25} &= 1 \\ \cos^2\theta &= \frac{21}{25} \\ \cos\theta &= \pm\frac{\sqrt{21}}{5} \end{aligned}$$

Ans $\boxed{\pm\frac{\sqrt{21}}{5}}$

Example 2: If $\sec\theta = -3$, what are the possible values of $\tan\theta$?

Solution

We use Pythagorean II: $1 + \tan^2\theta = \sec^2\theta$

Thus,

$$\begin{aligned} 1 + \tan^2\theta &= (-3)^2 \\ 1 + \tan^2\theta &= 9 \\ \tan^2\theta &= 8 \\ \tan\theta &= \pm\sqrt{8} = \pm 2\sqrt{2} \end{aligned}$$

Ans $\boxed{\pm 2\sqrt{2}}$
