4.5 Exponential and Logarithmic Models

A. Growth and Decay Models

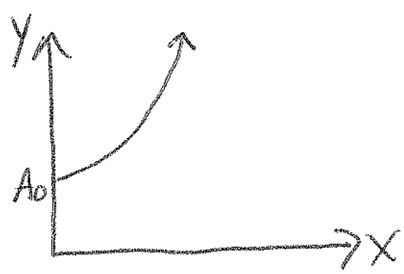
1. Growth and Decay Phenomena can be modeled by $A = A_0 e^{kt}$

   kt ← time
   k ← growth/decay rate
   A_0 ← initial amount
   A ← later amount

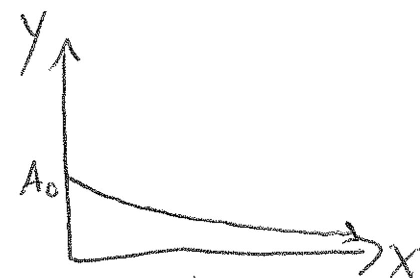
2. Model not derivable in this course. See MTH 235: Differential Equations.

3. Growth: $k > 0$
   Decay: $k < 0$

4.



$k > 0$
growth



$k < 0$
decay

5. Example Processes
   a. reproduction of bacteria (growth)
   b. radioactive decay of nuclei (decay)

6. Terminology
   a. doubling time: in a growth model, the time it takes for the quantity to grow to twice the initial amount
   b. half-life: in a decay model, time to decay to half the original amount

B. Solving Growth and Decay Problems

These are problems that "sort of" need to be solved <u>twice</u>.

<u>Method</u>: 1. Use the given information (usually first half of problem statement) to find and determine k for the model.

2. Start over with k plugged in and use the remaining information to answer the question.

C. Examples

<u>Example1</u>: A rabbit population grows exponentially. The population doubles every 40 days. Approximately how many rabbits are there at 90 days if the initial population is 10 rabbits?

<u>Solution</u>

1. First find k: $A = A_0 e^{kt}$

$$2A_0 = A_0 e^{k \cdot 40}$$

$$2 = e^{40k} \quad (\text{since } A_0 \neq 0)$$

$$\ln 2 = \ln(e^{40k}) = 40k$$

$$k = \frac{\ln 2}{40}$$

2. Now solve problem: $A = A_0 e^{\frac{\ln 2}{40} t}$

$$A = 10 e^{\frac{\ln 2}{40} \cdot 90} \approx 47.6$$

$\boxed{\text{Ans: Approximately 48 rabbits}}$

Example 2: ^{90}Sr decays radioactively with a half-life of 28 years. How long will it take for a 100g sample to decay to 15g?

Solution

1. Find k: $A = A_0 e^{kt}$

$$\frac{1}{2}A_0 = A_0 e^{k(28)}$$

$$\frac{1}{2} = e^{28k}$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{28k}\right) = 28k$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{28}$$

Note: $k<0$!

2. Now solve problem: $A = A_0 e^{\frac{\ln\left(\frac{1}{2}\right)}{28}t}$

$$15 = 100e^{\frac{\ln\left(\frac{1}{2}\right)}{28}t}$$

Now solve for t: $\frac{15}{100} = e^{\frac{\ln\left(\frac{1}{2}\right)}{28}t}$

$$\ln\left(\frac{15}{100}\right) = \cancel{\ln}\left(\cancel{e}^{\frac{\ln\left(\frac{1}{2}\right)}{28}t}\right) = \frac{\ln\left(\frac{1}{2}\right)}{28}t$$

$$\Rightarrow t = \frac{28\ln\left(\frac{15}{100}\right)}{\ln\left(\frac{1}{2}\right)} \approx 76.6 \text{ years}$$

Ans: $\boxed{\approx 76.6 \text{ years}}$

D. Newton's Law of Heating and Cooling

1. Law: A heated/cooled object placed in some different ambient temperature changes its temperature at a rate proportional to its temperature difference.

2. Model: $T = A + (T_0 - A)e^{-kt}, \ k > 0$

   T_0 = initial object temperature
   A = ambient temp

3. Model is not derivable in this course.

4. Examples
   a. hot cup of coffee cooling in a room
   b. potato heating in an oven
   c. can of pop cooling in a refrigerator
   d. cadaver cooling in room temperature

5. Solve problems using the same strategy as before: Find k, then answer question.

6. <u>Example Problem</u>: [redacted] A bottle of juice has a temperature of 70°F and is left to cool in a refrigerator at temperature of 45°F. After 10 minutes, the temperature of the juice is 55°F.

a) What is the temperature of the juice after 15 minutes?

b) When will the temperature of the juice be 50°F?

<u>Solution</u>

1. Find k: $T = A + (T_0 - A)e^{-kt}$

$$55 = 45 + (70-45)e^{-k(10)}$$

$$10 = 25e^{-10k}$$

$$\frac{10}{25} = e^{-10k}$$

$$\ln\left(\frac{10}{25}\right) = \ln\left(e^{-10k}\right) = -10k$$

$$k = \frac{\ln\left(\frac{10}{25}\right)}{-10}$$

<u>Note</u>: $k > 0$

a) $T = 45 + (70-45)e^{\frac{\ln\left(\frac{10}{25}\right)}{10}t}$

$$T = 45 + 25e^{\frac{\ln\left(\frac{10}{25}\right)}{10}t} \qquad (1)$$

Find T when $t = 15$:

$$T = 45 + 25e^{\frac{\ln\left(\frac{10}{25}\right)}{10}\cdot 15} \approx \boxed{51.3°F}$$

b) Using (1): $T = 45 + 25e^{\frac{\ln\left(\frac{10}{25}\right)}{10}t}$

$$50 = 45 + 25e^{\frac{\ln\left(\frac{10}{25}\right)}{10}t}$$

Now solve for t:

$$5 = 25e^{\frac{\ln\left(\frac{10}{25}\right)}{10}t}$$

$$\frac{5}{25} = e^{\frac{\ln\left(\frac{10}{25}\right)}{10}t}$$

$$\ln\left(\frac{1}{5}\right) = \cancel{\ln}\left(\cancel{e}^{\frac{\ln\left(\frac{10}{25}\right)}{10}t}\right)$$

$$\ln\left(\frac{1}{5}\right) = \frac{\ln\left(\frac{10}{25}\right)}{10}t$$

$$\Rightarrow t = \frac{10\ln\left(\frac{1}{5}\right)}{\ln\left(\frac{10}{25}\right)} \approx 17.6$$

<u>Ans</u>: $\boxed{\approx 17.6 \text{ min}}$