4.4B Logarithmic Equations

A. Logarithmic Equation Strategy

1. Condense logarithms, so there is at most one logarithm on each side. Make sure they are the same base!
2. Exponentiate to kill the logarithm.
3. Solve the remaining equation.
4. Important: Check the solution(s) in the original equation to make sure that you don't have a log of a negative number!

B. Logarithmic Equation Examples

Example 1: Solve $\log_4(1-x)=3$ for x

Solution

$$\log_4(1-x)=3$$

Apply $4^{\square}$ to each side to kill $\log_4$!

$$4^{\log_4(1-x)}=4^3$$

$$1-x=64$$

$$x=1-64=-63$$

Need to check!: $\log_4(1-(-63))=\log_4 64$ ok ✓

Ans: $\boxed{\{-63\}}$

Example 2: Solve $\ln(4x) - 3\ln(x^2) = \ln 2$ for x

Solution

Condense! $\ln(4x) - 3\ln(x^2) = \ln 2$

$\ln(4x) - \ln[(x^2)^3] = \ln 2$ Undo Power Rule

$\ln(4x) - \ln(x^6) = \ln 2$

$\ln\left(\frac{4x}{x^6}\right) = \ln 2$ Undo Quotient Rule

Apply $e^{\square}$ to each side to kill $\ln$ ($\log_e$):

$$e^{\ln\left(\frac{4}{x^5}\right)} = e^{\ln 2}$$

$$\frac{4}{x^5} = 2$$

$$x^5 = 2 \Rightarrow x = \sqrt[5]{2}$$

Need to do a check!

$\ln(4\sqrt[5]{2}) - 3\ln((\sqrt[5]{2})^2)$ ✓

positive: is ok

Ans: $\boxed{\sqrt[5]{2}}$

Example 3: Solve $\log_4(\log_4 x) = 1$ for x

Solution

Apply $4^{\square}$ to each side to kill "outer" $\log_4$:

$$\log_4(\log_4 x) = 1$$

$$\cancel{4^{\log_4}}{}^{(\log_4 x)} = 4^1$$

$$\log_4 x = 4$$

Apply $4^{\square}$ to each side to kill $\log_4$:

$$\cancel{4^{\log_4}}{}^{x} = 4^4$$

$$x = 4^4 = 256$$

Need to do a check! $\log_4(\log_4 256)$

$= \log_4(\cancel{\log_4 4}^4)$

$= \log_4 4$

positive: OK

Ans: $\boxed{\{256\}}$

Example 4: Solve $\log_3(x-4) = \log_9(2x+1)$ for x

Solution

Not the same base: use the change of base formula

$$\log_3(x-4) = \log_9(2x+1)$$

$$\Rightarrow \log_3(x-4) = \frac{\log_3(2x+1)}{\log_3 9}$$

$$\Rightarrow \log_3(x-4) = \frac{\log_3(2x+1)}{\cancel{\log_3} 3^2}$$

$$\Rightarrow \log_3(x-4) = \frac{\log_3(2x+1)}{2}$$

$$\Rightarrow 2\log_3(x-4) = \log_3(2x+1)$$

Condense:

$$\log_3(x-4)^2 = \log_3(2x+1) \qquad \text{Undo Power Rule}$$

Apply $3^{\square}$ to each side to kill $\log_3$:

$$\cancel{3^{\log_3}}(x-4)^2 = \cancel{3^{\log_3}}(2x+1)$$

$$(x-4)^2 = (2x+1)$$

$$x^2-8x+16=2x+1$$

$$x^2-10x+15=0$$

QF: $x=\dfrac{10\pm\sqrt{100-4(1)(15)}}{2(1)}$

$$x=\frac{10\pm\sqrt{40}}{2}=\frac{10\pm2\sqrt{10}}{2}$$

$$x=\frac{2(5\pm\sqrt{10})}{2}=5\pm\sqrt{10}$$

Need to do a check!

$5-\sqrt{10}$? $\quad \log_3(x-4)=\log_9(2x+1)$

$$\log_3(5-\sqrt{10}-4)\stackrel{?}{=}\log_9(2(5-\sqrt{10})+1)$$

$$\log_3(1-\sqrt{10})\stackrel{?}{=}\log_9(11-2\sqrt{10})$$

negative!

X

$5+\sqrt{10}$? $\quad \log_3(x-4)=\log_9(2x+1)$

$$\log_3(5+\sqrt{10}-4)\stackrel{?}{=}\log_9(2(5+\sqrt{10})+1)$$

$$\log_3(1+\sqrt{10})\stackrel{?}{=}\log_9(11+2\sqrt{10})$$

both positive

OK!

Ans: $\boxed{\{5+\sqrt{10}\}}$

C. Compound Interest Revisited

1. Formulas: Compounding n times/yr: $A=P\left(1+\frac{r}{n}\right)^{nt}$

   Compounding continuously: $A=Pe^{rt}$

2. Some compound interest problems require logarithms.

3. Example: How long will it take for an initial investment to double if the savings account pays 2%/yr compounded monthly?

Solution

Use $A=P\left(1+\frac{.02}{12}\right)^{12t}$

We want to know when $A=2P$ ...

$$2P=P\left(1+\frac{.02}{12}\right)^{12t}$$

Divide by P $[P\neq 0]$: $2=\left(1+\frac{.02}{12}\right)^{12t}$

Thus $\ln 2=\ln\left(1+\frac{.02}{12}\right)^{12t}$

$$\ln 2 = 12t\ln\left(1+\frac{.02}{12}\right)$$

$$\Rightarrow t=\frac{\ln 2}{12\ln\left(1+\frac{.02}{12}\right)}\approx \boxed{34.7\text{ years}}$$