# 4.4A Exponential Equations

## A. Exponential Equation Strategy

1. Isolate the exponential.
2. See if you can solve using the base property. If not, take natural logs "ln" of each side

   Note: We take "ln" rather than the "obvious log" to avoid having to use the change-of-base formula.
3. Use the power rule (in reverse) to get the variable out of the exponent spot.
4. Solve the remaining equation.

## B. Examples

Example 1: Solve $3^{x+2}=5$ for $x$

Solution

$$3^{x+2}=5$$

$$\ln(3^{x+2})=\ln 5$$

Power Rule (in reverse): $(x+2)\ln 3=\ln 5$

$$x\ln 3+2\ln 3=\ln 5$$

$$x\ln 3=\ln 5-2\ln 3$$

$$x=\frac{\ln 5-2\ln 3}{\ln 3}$$

Ans: $\boxed{\left\{\frac{\ln 5-2\ln 3}{\ln 3}\right\}}$

Example 2: Solve $2(7^{2x-1})+3=9$ for $x$

Solution

Isolate the exponential:

$$2(7^{2x-1})=6$$

$$7^{2x-1}=3$$

Take natural logs:

$$\ln(7^{2x-1})=\ln 3$$

Power Rule: (in reverse) $(2x-1)\ln 7=\ln 3$

$$2x\ln 7-\ln 7=\ln 3$$

$$2x\ln 7=\ln 3+\ln 7$$

$$x=\frac{\ln 3+\ln 7}{2\ln 7}$$

Ans: $\boxed{\left\{\frac{\ln 3+\ln 7}{2\ln 7}\right\}}$

Example 3: Solve $4^{3x-2} = 7^{x+1}$ for x

Solution

Take natural logs:

$$\ln\left(4^{3x-2}\right) = \ln\left(7^{x+1}\right)$$

Power rule (in reverse):

$$(3x-2)\ln 4 = (x+1)\ln 7$$

$$3x\ln 4 - 2\ln 4 = x\ln 7 + \ln 7$$

Isolate x:

$$3x\ln 4 - x\ln 7 = 2\ln 4 + \ln 7$$

$$x(3\ln 4 - \ln 7) = 2\ln 4 + \ln 7$$

$$x = \frac{2\ln 4 + \ln 7}{3\ln 4 - \ln 7}$$

Ans: $\boxed{\left\{\frac{2\ln 4 + \ln 7}{3\ln 4 - \ln 7}\right\}}$

Example 4: Solve $\begin{cases} 3^{2x+y} = 7 \\ 5^{x-y} = 6 \end{cases}$ for $x$ and $y$

Solution

Use $\ln$ on each:

$$\begin{cases} \ln(3^{2x+y}) = \ln 7 \\ \ln(5^{x-y}) = \ln 6 \end{cases}$$

$$\Rightarrow \begin{cases} (2x+y)\ln 3 = \ln 7 \\ (x-y)\ln 5 = \ln 6 \end{cases}$$

$$\Rightarrow \begin{cases} 2x\ln 3 + y\ln 3 = \ln 7 & (1) \\ x\ln 5 - y\ln 5 = \ln 6 & (2) \end{cases}$$

Use standard elimination techniques to solve for $x$ and $y$:

Eliminate $x$:

Multiply (1) by $\ln 5$ and (2) by $-2\ln 3$:

$$\begin{array}{rcl} 2x\ln 3\ln 5 + y\ln 3\ln 5 &=& \ln 5\ln 7 \\ -2x\ln 3\ln 5 + 2y\ln 3\ln 5 &=& -2\ln 3\ln 6 \\ \hline \end{array} \quad \text{ADD}$$

$$3y\ln 3\ln 5 = \ln 5\ln 7 - 2\ln 3\ln 6$$

$$y = \frac{\ln 5 \ln 7 - 2\ln 3 \ln 6}{3\ln 3 \ln 5}$$

Eliminate $y$:

Multiply (1) by $\ln 5$ and (2) by $\ln 3$:

$$\begin{array}{r} 2x\ln 3\ln 5 + y\ln 3\ln 5 = \ln 5\ln 7 \\ x\ln 3\ln 5 - y\ln 3\ln 5 = \ln 3\ln 6 \\ \hline 3x\ln 3\ln 5 = \ln 3\ln 6 + \ln 5\ln 7 \end{array}$$

ADD

$$x = \frac{\ln 3\ln 6 + \ln 5\ln 7}{3\ln 3\ln 5}$$

Ans: $\boxed{(x,y) = \left(\dfrac{\ln 3\ln 6 + \ln 5\ln 7}{3\ln 3\ln 5}, \dfrac{\ln 5\ln 7 - 2\ln 3\ln 6}{3\ln 3\ln 5}\right)}$

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Example 5: Solve $\dfrac{e^x + e^{-x}}{2} = 7$ for $x$

Solution

$$\frac{e^x + e^{-x}}{2} = 7 \Rightarrow e^x + e^{-x} = 14$$

$$e^x + \frac{1}{e^x} = 14$$

$$e^x\left(e^x + \frac{1}{e^x}\right) = e^x(14)$$

$$(e^x)^2 + 1 = 14e^x$$

$$(e^x)^2 - 14(e^x) + 1 = 0$$

Quadratic in Form

Clever Substitution $u = e^x$

$$u^2 - 14u + 1 = 0$$

<u>QF:</u> $u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{14 \pm \sqrt{14^2 - 4(1)(1)}}{2(1)}$

$$u = \frac{14 \pm \sqrt{192}}{2} = \frac{14 \pm 8\sqrt{3}}{2} = \frac{2(7 \pm 4\sqrt{3})}{2}$$

$$u = 7 \pm 4\sqrt{3}$$

Thus $e^x = 7 \pm 4\sqrt{3}$

$$\ln(e^x) = \ln(7 \pm 4\sqrt{3})$$

↑ Neither is negative, so both are OK!

$$x = \ln(7 \pm 4\sqrt{3})$$

<u>Ans:</u> $\boxed{\{\ln(7 - 4\sqrt{3}), \ln(7 + 4\sqrt{3})\}}$