4.1 D Compound Interest

A. Development: Discrete Case

Invest Principal P into a bank account paying an annual interest rate, r, and compounded n times per year

1. Initial Investment: P
2. Compounded Monthly: Monthly Interest Rate $\frac{r}{12}$
3. Interest After 1 month: $P \cdot \frac{r}{12}$
4. New Balance After 1 month: $P + P \cdot \frac{r}{12} = P\left(1+\frac{r}{12}\right)$
5. New Balance After 2 months: $P\left(1+\frac{r}{12}\right)\left(1+\frac{r}{12}\right) = P\left(1+\frac{r}{12}\right)^2$
6. New Balance After 36 months: $P\left(1+\frac{r}{12}\right)^{36} = P\left(1+\frac{r}{12}\right)^{12 \cdot 3}$
7. New Balance After 17 years: $P\left(1+\frac{r}{12}\right)^{12 \cdot 17}$
8. New Balance After t years: $P\left(1+\frac{r}{12}\right)^{12t}$
9. Compounding n times per year rather than 12: $P\left(1+\frac{r}{n}\right)^{nt}$

B. Discrete Compounding Interest Formula

$$\boxed{A = P\left(1+\frac{r}{n}\right)^{nt}}$$

C. Development: Continuous Case

We consider what happens if you let the number of times you compound, $n \to \infty$

1. Discrete Formula: $A = P\left(1+\frac{r}{n}\right)^{nt}$ $\quad (*)$

2. Let $m = \frac{n}{r}$. Note that $m \to \infty$ when $n \to \infty$

3. Then $n = mr$ and substituting into $(*)$:

$$A = P\left(1+\frac{r}{mr}\right)^{mrt}$$

$$A = P\left(1+\frac{1}{m}\right)^{mrt} \quad (**)$$

4. Continuous Compounding

Let $m \to \infty$ in $(**)$:

$$A = P\left[\lim_{m \to \infty}\left(1+\frac{1}{m}\right)^{m}\right]^{rt}$$

$$A = Pe^{rt}$$

D. Continuous Compounding Interest Formula

$$\boxed{A = Pe^{rt}}$$

E. Example

After 3 years, what is the balance on a savings account with an initial investment of \$2000, if the account pays

a) 2.7% compounded quarterly
b) 2.7% compounded continuously

Solution

a) quarterly $\Rightarrow$ use discrete model with $n=4$

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$= 2000\left(1+\frac{.027}{4}\right)^{4(3)} \approx \boxed{\$2168.15}$$

b) continuously $\Rightarrow$ use continuous model

$$A = Pe^{rt}$$

$$= 2000\,e^{(.027)3} \approx \boxed{\$2168.74}$$