3.6 A Polynomial Inequalities — covered after Ch 1

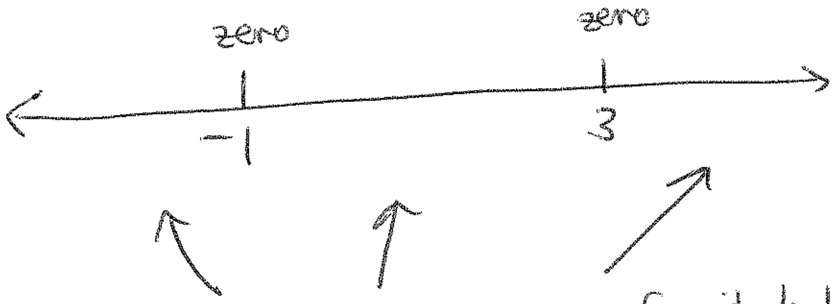
A. Introduction

Suppose we want to solve $(x+1)(3-x)>0$ for x

We know when the left hand side is zero (by ZPP), namely at $x=-1$ and $x=3$.

We want where it is positive (bigger than zero).

It can only happen in three possible areas:



Three possible places for it to be bigger than zero.

We just need to check these three zones to figure out where it is positive:

Test Zone I: Plug in any number smaller than -1!

Plug in $x=-2$:

$$(-2+1)(3-(-2))=(-1)(5)=-5 \qquad -$$

Test Zone II: Plug in any number between -1 and 3

Try $x=0$:

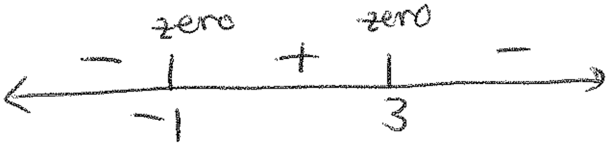
$$(0+1)(3-0)=(1)(3)=3 \qquad +$$

Test Zone III: Plug in any number bigger than 3:

Try $x=4$:

$$(4+1)(3-4)=(5)(-1)=-5 \qquad -$$

Thus for $(x+1)(3-x)$, we have the data:



We want $(x+1)(3-x) > 0$.

Thus from our number line, we have $\boxed{(-1,3)}$ Ans

B. Strategy

With right hand side zero, and left-hand side factored:

1. Use ZPP to determine where the expression is zero. These are called the breakpoints.

2. Put breakpoints on a number line. Label breakpoints as "zero
3. Pick a point from each zone to determine the "sign" of the zon
4. Read answer off of the signed number line.

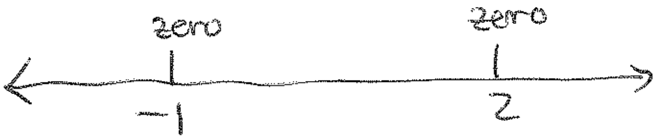
C. Examples

Example 1: Solve $(x+1)^2(x-2) \geq 0$ for x

Solution

1. Breakpoints: $x+1=0 \Rightarrow x=-1$
$x-2=0 \Rightarrow x=2$

2.



3. Test Zone I: Pick $x=-2$: $(-2+1)^2(-2-2)$

$= (-1)^2(-4)$

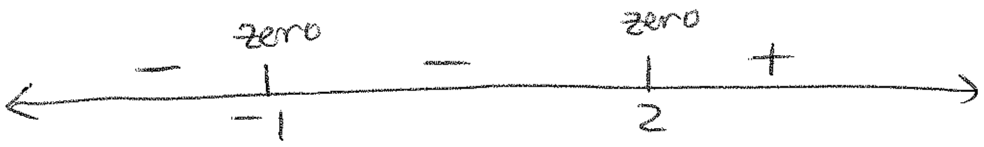
$= (1)(-4) = -4$ $\quad -$

Test Zone II: Pick $x=0$: $(0+1)^2(0-2)$

$= 1^2(-2) = -2$ $\quad -$

Test Zone III: Pick $x=3$: $(3+1)^2(3-2)$

$= 4^2 \cdot 1 = 16$ $\quad +$



4. We want ≥ 0, so we want where it is zero and where it is $+$.

Ans: $\boxed{\begin{array}{c} \{-1\} \cup [2,\infty) \\ \text{-or-} \\ [-1,-1] \cup [2,\infty) \end{array}}$

Example 2: Solve $x^3 < 4x$ for x

Solution

Move to one side: $x^3 - 4x < 0$

Factor: $x(x^2-4) < 0$

$x(x+2)(x-2) < 0$ $\quad (*)$

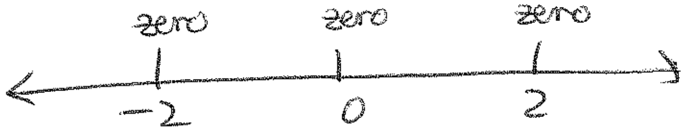
Now use strategy ...

1. <u>Breakpoints</u>: $x=0$

$x+2=0 \Rightarrow x=-2$

$x-2=0 \Rightarrow x=2$

2.



3. <u>Test Zones</u>:

<u>Zone I</u>: Pick $x=-3$: (using $x(x+2)(x-2)$)

$(-3)(-3+2)(-3-2)=(-3)(-1)(-5)$ $-$

<u>Zone II</u>: Pick $x=-1$:

$(-1)(-1+2)(-1-2)=(-1)(1)(-3)$ $+$

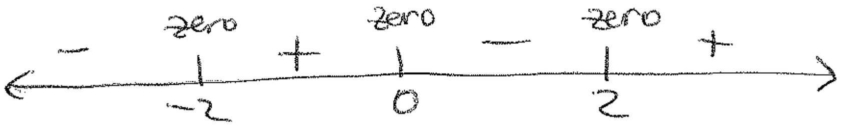
<u>Zone III</u>: Pick $x=1$:

$1(1+2)(1-2) = 1(3)(-1)$ $-$

<u>Zone IV</u>: Pick $x=3$:

$3(3+2)(3-2)= 3(5)(1)$ $+$

4.



Want <0 by (*)

<u>Ans</u>: $\boxed{(-\infty,-2)\cup(0,2)}$

Example 3: Solve $(2x^2-5x-12)(x^2+5x+1)(x^2+x+1) \geq 0$ for x

Solution

Factor left hand side:

① $$\begin{array}{l|l} \underline{2x^2-5x-12} & \underline{\textcircled{-24}\ +,-} \\ 2x^2+x-6x-12 & -6 \\ 2x^2+2x-7x-12 & -14 \\ 2x^2+3x-8x-12 & -24 \ \checkmark \\ x(2x+3)-4(2x+3) & \\ (2x+3)(x-4) & \end{array}$$

② $$\begin{array}{l|l} \underline{x^2+5x+1} & \underline{\textcircled{1}\ +,+} \\ x^2+x+4x+1 & 4 \\ x^2+2x+3x+1 & 6 \ \times \end{array}$$

prime

③ x^2+x+1 prime

$$(2x+3)(x-4)(x^2+5x+1)(x^2+x+1) \geq 0 \qquad (*)$$

Now use strategy:

1. Breakpoints: $2x+3=0 \Longrightarrow x=-\frac{3}{2}$

$x-4=0 \Longrightarrow x=4$

$x^2+5x+1=0$ prime, so use QF

$$x=\frac{-5\pm\sqrt{25-4(1)(1)}}{2(1)}=\frac{-5\pm\sqrt{21}}{2}$$

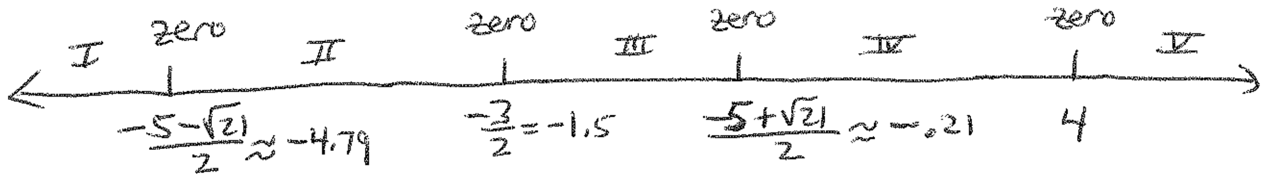
$x^2+x+1=0$ prime, so use QF

$$x=\frac{-1\pm\sqrt{1^2-4(1)(1)}}{2(1)}=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}$$

$$=\frac{-1\pm i\sqrt{3}}{2} \Leftarrow$$ not real, so no breakpoints

Breakpoints: $-\frac{3}{2}, 4, \frac{-5-\sqrt{21}}{2} \approx -4.79, \frac{-5+\sqrt{21}}{2} \approx -.21$

2.



3. <u>Test Zones</u>: Use (*): $(2x+3)(x-4)(x^2+5x+1)(x^2+x+1)$

<u>Test I</u>: $x=-5$:

$$[2(-5)+3][-5-4][(-5)^2+5(-5)+1][(-5)^2+(-5)+1]$$

$$=(-7)(-9)(1)(16) \qquad +$$

<u>Test II</u>: $x=-2$:

$$[2(-2)+3][-2-4][(-2)^2+5(-2)+1][(-2)^2+(-2)+1]$$

$$=(-1)(-6)(-5)(3) \qquad -$$

<u>Test III</u>: $x=-1$:

$$[2(-1)+3][-1-4][(-1)^2+5(-1)+1][(-1)^2+(-1)+1]$$

$$=(1)(-5)(-3)(1) \qquad +$$

<u>Test IV</u>: $x=0$:

$$[2(0)+3][0-4][0^2+5(0)+1][0^2+0+1]$$

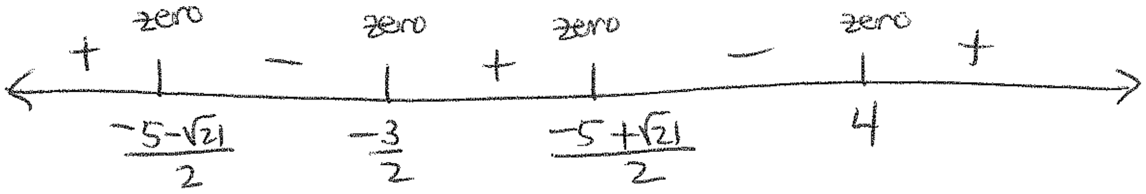
$$=(3)(-4)(1)(1) \qquad -$$

<u>Test V</u>: $x=5$:

$$[2(5)+3][5-4][5^2+5(5)+1][5^2+5+1]$$

$$(13)(1)(51)(31) \qquad +$$

4.



Since by (*), $(2x+3)(x-4)(x^2+5x+1)(x^2+x+1) \geq 0$, we want the zones that are positive <u>or</u> zero

<u>Ans</u>: $\boxed{\left(-\infty, \frac{-5-\sqrt{21}}{2}\right] \cup \left[-\frac{3}{2}, \frac{-5+\sqrt{21}}{2}\right] \cup [4, \infty)}$

D. Comments

1. The zones don't always alternate, you have to check them all. [e.g. Example 1]

2. One side must always be zero before using the breakpoint idea

EX) $(x+1)(x-3) > 6$

You <u>have to</u> multiply everything out, move the 6 over, refactor

3. No matter how simple an inequality looks, unless linear or absolute value, you <u>must</u> solve using the breakpoint method

For instance, if you have powers, you <u>must</u> use breakpoints

EX) Solve $x^2 \geq 4$ for x

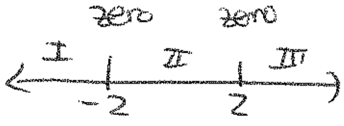
<u>Solution</u>

Square root principle <u>NOT VALID</u> for inequalities!
must do the following:

$$x^2 - 4 \geq 0$$

$$(x+2)(x-2) \geq 0$$

Breakpoints: $-2, 2$



<u>etc.</u> !!!