3.5E Algebraic Functions

A. Introduction

The next most complicated type of function to study is to "take rational functions and throw roots into the mix".

For example, $f(x)=\frac{\sqrt{x-3}}{x^2-4}$

The way to describe these "algebraic functions" formally is to clear fractions, clear roots, and move everything to one side.

$$y=\frac{\sqrt{x-3}}{x^2-4}$$

$$(x^2-4)\,y=\sqrt{x-3}$$

$$(x^2-4)^2y^2=x-3$$

$$(x^4-8x^2+16)y^2=x-3$$

$$(x^4-8x^2+16)y^2-(x-3)=0$$

This looks like a polynomial <u>in y</u>, whose "coefficients" are polynomials in x.

We take this model as the definition of an algebraic function.

B. Definition of an Algebraic Function

An algebraic function f is one whose output formula $y = f(x)$ may be rewritten as

$$p_n(x)y^n + p_{n-1}(x)y^{n-1} + \dots + p_1(x)y + p_0(x) = 0$$

i.e. a polynomial in y, whose "coefficients" are polynomials in x

C. Method for Showing a Function is Algebraic

By repeatedly clear fractions and clearing roots, show that you get an equation like above after moving everything to one side.

D. Examples

Example 1: Show that $f(x) = \sqrt[3]{\frac{x-1}{x^2+3}}$ defines f as an algebraic function.

Solution

$$y = \sqrt[3]{\frac{x-1}{x^2+3}} \Rightarrow y^3 = \frac{x-1}{x^2+3} \Rightarrow (x^2+3)y^3 = x-1$$

Thus $\boxed{(x^2+3)y^3 - (x-1) = 0}$ Ans.

Note: Here $p_3(x) = x^2+3$
$p_0(x) = -(x-1)$

Example 2: Show that $f(x)=\sqrt{x}+\frac{1}{\sqrt{x}}+1$ defines f as an algebraic function.

Solution

$$y=\sqrt{x}+\frac{1}{\sqrt{x}}+1$$

Clear fractions: $\sqrt{x}\cdot y=\sqrt{x}\left(\sqrt{x}+\frac{1}{\sqrt{x}}+1\right)$

$$y\sqrt{x}=x+1+\sqrt{x}$$

Isolate radical:

$$y\sqrt{x}-\sqrt{x}=x+1$$

$$\sqrt{x}(y-1)=x+1$$

$$\sqrt{x}=\frac{x+1}{y-1}$$

Clear radical:

$$x=\left(\frac{x+1}{y-1}\right)^2$$

$$x=\frac{x^2+2x+1}{y^2-2y+1}$$

Clear fractions: $x(y^2-2y+1)=x^2+2x+1$

$$xy^2-2xy+x=x^2+2x+1$$

Move everything to one side:

$$xy^2 - 2xy - x^2 - x - 1 = 0$$

$$\boxed{xy^2 - 2xy - (x^2+x+1) = 0} \quad \underline{\text{Ans}}$$

<u>Note</u>: $p_2(x) = x$
$p_1(x) = -2x$
$p_0(x) = -(x^2+x+1)$

E. Graphing Algebraic Functions

As you can imagine, this is even more complicated than graphing rational functions! We will skip this, as some tools in calculus make this easier.

F. End of Algebra

1. <u>Algebra</u> is the study of algebraic functions.

2. Since we now go beyond the study of algebraic functions, we are leaving algebra!

3. Functions that are <u>not</u> algebraic are called <u>transcendental functions</u>.

4. Some popular transcendental functions to study before starting calculus are
   ① Exponential functions
   ② Logarithmic functions
   ③ Trigonometric functions