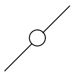
3.5C Rational Functions and Asymptotes

A. Definition of a Rational Function

f is said to be a **rational function** if $f(x) = \frac{g(x)}{h(x)}$, where g and h are polynomial functions. That is, rational functions are fractions with polynomials in the numerator and denominator.

B. Asymptotes/Holes

Holes are what they sound like:

 is a hole

Rational functions may have holes or asymptotes (or both!).

Asymptote Types:

1. vertical
2. horizontal
3. oblique ("slanted-line")
4. curvilinear (asymptote is a curve!)

We will now discuss how to find all of these things.

C. Finding Vertical Asymptotes and Holes

Factors in the denominator cause vertical asymptotes and/or holes.

To find them:

1. Factor the denominator (and numerator, if possible).

2. Cancel common factors.

3. Denominator factors that **cancel** completely give rise to **holes**. Those that don't give rise to **vertical asymptotes**.

D. Examples

Example 1: Find the vertical asymptotes/holes for f where $f(x) = \frac{(3x+1)(x-7)(x+4)}{(x-7)^2(x+4)}$.

Solution

Canceling common factors: $f(x) = \frac{3x+1}{x-7}$, $x \neq -4$

$x + 4$ factor cancels completely $\Rightarrow$ **hole** at $x = -4$

$x - 7$ factor not completely canceled $\Rightarrow$ **vertical asymptote** with equation $x = 7$

Example 2: Find the vertical asymptotes/holes for f where $f(x) = \frac{2x^2-5x-12}{x^2-5x+4}$.

Solution

Factor: $f(x) = \frac{(x-4)(2x+3)}{(x-4)(x-1)}$

Cancel: $f(x) = \frac{2x+3}{x-1}$, $x \neq 4$

Ans **Hole** at $x = 4$
Vertical Asymptote with equation $x = 1$

E. Finding Horizontal, Oblique, Curvilinear Asymptotes

Suppose $f(x) = \dfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$

If

1. degree top $<$ degree bottom: **horizontal asymptote** with equation $y = 0$
2. degree top $=$ degree bottom: **horizontal asymptote** with equation $y = \frac{a_n}{b_m}$
3. degree top $>$ degree bottom: **oblique** or **curvilinear asymptotes**

To find them: Long divide and throw away remainder

F. Examples

Example 1: Find the horizontal, oblique, or curvilinear asymptote for f where $f(x) = \frac{6x^4 - x + 2}{7x^5 + 2x - 1}$.

Solution

degree top$= 4$ degree bottom$= 5$. Since $4 < 5$, we have

Ans **horizontal asymptote** with equation $y = 0$

Example 2: Find the horizontal, oblique, or curvilinear asymptote for f where $f(x) = \frac{6x^3-2x^2+1}{2x^3+5}$.

Solution

degree top$= 3$ degree bottom$= 3$.

Since $3 = 3$, we have a horizontal asymptote of $y = \frac{6}{2} = 3$. Thus

Ans **horizontal asymptote** with equation $y = 3$

Example 3: Find the horizontal, oblique, or curvilinear asymptote for f where $f(x) = \frac{2x^3-3}{x^2-1}$.

Solution

degree top$= 3$ degree bottom$= 2$.

Since $3 > 2$, we have an oblique or curvilinear asymptote. Now long divide:

$$\begin{array}{r|l} & 2x \\ \hline x^2 + 0x - 1 & 2x^3 + 0x^2 + 0x - 3 \\ & -(2x^3 + 0x^2 - 2x) \\ \hline & \qquad\qquad 2x - 3 \end{array}$$

Since $\dfrac{2x^3-3}{x^2-1} = 2x + \underbrace{\dfrac{2x-3}{x^2-1}}_{\text{Throw away}}$, we have that

Ans $y = 2x$ defines a line, and is the equation for the **oblique asymptote**

Example 4: Find the horizontal, oblique, or curvilinear asymptote for f where

$$f(x) = \frac{3x^5 - x^4 + 2x^2 + x + 1}{x^2 + 1}.$$

Solution

degree top$= 5$ degree bottom$= 2$.

Since $5 > 2$, we have an oblique or curvilinear asymptote. Now long divide:

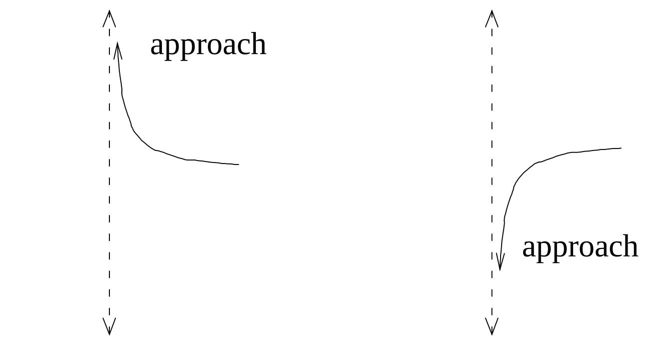
$$\begin{array}{r|l} & 3x^3 - x^2 - 3x + 3 \\ \hline x^2 + 0x + 1 & 3x^5 - x^4 + 0x^3 + 2x^2 + x + 1 \\ & -\underline{(3x^5 + 0x^4 + 3x^3)} \\ & \qquad -x^4 - 3x^3 + 2x^2 \\ & \qquad -\underline{(-x^4 + 0x^3 - x^2)} \\ & \qquad\qquad -3x^3 + 3x^2 + x \\ & \qquad\qquad -\underline{(-3x^3 + 0x^2 - 3x)} \\ & \qquad\qquad\qquad 3x^2 + 4x + 1 \\ & \qquad\qquad\qquad -\underline{(3x^2 + 0x + 3)} \\ & \qquad\qquad\qquad\qquad 4x - 2 \end{array}$$

Since $\dfrac{3x^5 - x^4 + 2x^2 + x + 1}{x^2 + 1} = 3x^3 - x^2 - 3x + 3 + \underbrace{\dfrac{4x - 2}{x^2 + 1}}_{\text{Throw away}}$, we have that

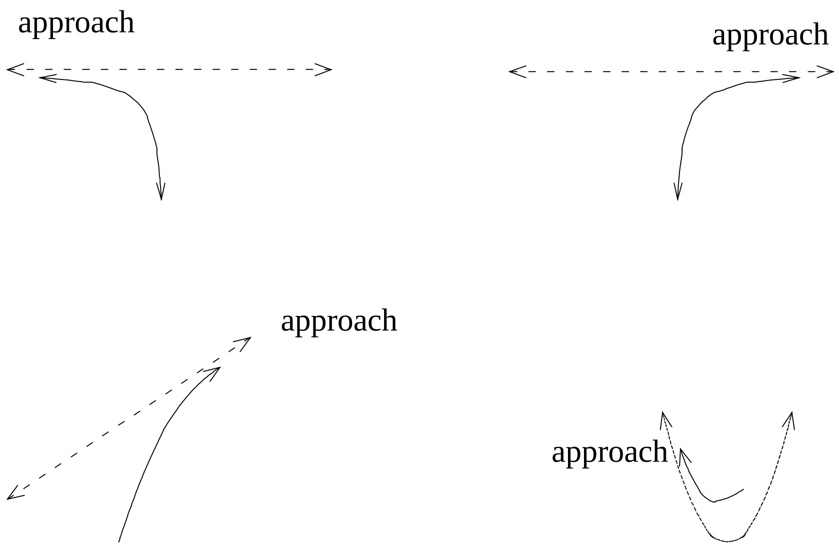
Ans $\boxed{y = 3x^3 - x^2 - 3x + 3 \text{ defines a } \textbf{curvilinear asymptote}}$

G. Asymptote Discussion for Functions

1. As the graph of a function approaches a **vertical asymptote**, it shoots up or down toward $\pm\infty$.



2. Graphs approach **horizontal, oblique,** and **curvilinear asymptotes** as $x \to -\infty$ or $x \to \infty$.



3. Graphs of functions **never** cross vertical asymptotes, but **may** cross other asymptote types.