# 3.4G Complex Zeros V: Ring Problem

## A. Motivating Example

Factor $x^4 - 25$:

1. Over the integers, $\mathbb{Z}$
2. Over the rationals, $\mathbb{Q}$
3. Over the reals, $\mathbb{R}$
4. Over the complex numbers, $\mathbb{C}$

Solution

1. over the integers:

$$x^4 - 25 = (x^2)^2 - 5^2 \quad \text{difference of squares}$$

$$= \boxed{(x^2+5)(x^2-5)} \quad \text{can't factor further}$$

2. Over the rationals: same answer: $\boxed{(x^2+5)(x^2-5)}$

3. over the reals: We can factor further, because by using roots we can view 5 as $(\sqrt{5})^2$!

$$(x^2+5)\underbrace{(x^2-(\sqrt{5})^2)}_{\text{difference of squares}}$$

$$\boxed{(x^2+5)(x+\sqrt{5})(x-\sqrt{5})} \quad \text{can't factor further}$$

This illustrates the Factoring Over Reals Theorem

(4) Over the complex numbers: We know by the Linear Factorization Theorem that $x^2+5$ can be broken into linear factors

Method 1: Find zeros / Factor Thm

$$x^2+5=0$$
$$x^2=-5$$
$$x=\pm\sqrt{-5}$$
$$x=\pm i\sqrt{5}$$

Factors: $(x-i\sqrt{5})(x+i\sqrt{5})$ ← conjugate pairs principle!

Method 2: Clever trick w/ difference of squares

$$x^2+5 = x^2-(-5)$$
$$= x^2-(\sqrt{-5})^2 \quad \text{difference of squares}$$
$$= (x+\sqrt{-5})(x-\sqrt{-5})$$
$$= (x+i\sqrt{5})(x-i\sqrt{5})$$

Thus the answer is

$$\boxed{(x+i\sqrt{5})(x-i\sqrt{5})(x+\sqrt{5})(x-\sqrt{5})}$$

## B. Comments on Example

Given $x^4-25$,

1. Over $\mathbb{Z}$, it factors as $(x^2+5)(x^2-5)$
2. Over $\mathbb{Q}$, it factors as $(x^2+5)(x^2-5)$
3. Over $\mathbb{R}$, it factors as $(x^2+5)(x+\sqrt{5})(x-\sqrt{5})$
4. Over $\mathbb{C}$, it factors as $(x+i\sqrt{5})(x-i\sqrt{5})(x+\sqrt{5})(x-\sqrt{5})$

This demonstrates that asking for something to be "factored" is ambivalent! There are different possible answers depending on "what you are factoring <u>over</u>"

In general, when you do factoring, you factor over <u>rings</u>. Thus $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are different rings you might factor over

Thus, a brief vocabulary lesson to follow...

## C. Some Vocabulary

1. A <u>ring</u> is a set where you can add, subtract, and multiply (and stay inside)

   EX) Natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$ is <u>not</u> a ring

   because $3-7=-4$ not in $\mathbb{N}$!

EX2) Are the whole numbers $\mathbb{W} = \{0, 1, 2, 3, ..\}$ a ring?

Ans: No: $3-7=-4$ is not in $\mathbb{W}$

EX3) $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are all rings

2. A field is a set where you can add, subtract, multiply, and divide (excepting 0)

EX1) $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$ are not fields

EX2) $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are fields

EX3) Every field is a ring, but not every ring is a field ($\mathbb{Z}$ is a ring, but not a field)

D. Closing Comment

The study of rings and fields in a more general setting is the basis of a course in abstract algebra (see MTH310)