3.4F Complex Zeros IV: Going Backward

A. Introduction

We consider examples where you are given some zeros of a polynomial function and they want you to reconstruct the original polynomial.

We use the Linear Factorization Theorem. The "a" is determined by a side condition.

B. Examples

Example 1: 2, −1, and $1+i$ are zeros of a polynomial function. Find the polynomial of lowest degree satisfying this, where the polynomial $f(x)$ has $f(1)=4$.

Solution

By the Conjugate Pairs Principle, $1-i$ is also a zero.

By the Linear Factorization Theorem, we have

$$f(x) = a(x-2)(x+1)[x-(1+i)]\cdot[x-(1-i)]$$

$$= a(x-2)(x+1)(x-1-i)(x-1+i)$$

$$= a(x-2)(x+1)[(x-1)-i][(x-1)+i]$$

$$= a(x-2)(x+1)[(x-1)^2-i^2]$$

$$= a(x-2)(x+1)(x^2-2x+1+1)$$

$$= a(x-2)(x+1)(x^2-2x+2)$$ ← illustrates Factoring over Reals Thm.

So $f(x) = a(x^2 - x - 2)(x^2 - 2x + 2)$

$$\begin{array}{r|l} & x^2 - x - 2 \\ \hline x^2 & x^4 - x^3 - 2x^2 \\ -2x & -2x^3 + 2x^2 + 4x \\ +2 & 2x^2 - 2x - 4 \end{array}$$

$$f(x) = a(x^4 - 3x^3 + 2x^2 + 2x - 4) \qquad (*)$$

Apply side condition: $f(1) = 4$:
(to find a)

$$f(1) = a[1^4 - 3(1)^3 + 2(1)^2 + 2(1) - 4] = 4$$

$$\Rightarrow \quad a(1 - 3 + 2 + 2 - 4) = 4$$

$$\Rightarrow \quad -2a = 4$$

$$\Rightarrow \quad a = -2$$

Substituting back into (*):

$$f(x) = -2(x^4 - 3x^3 + 2x^2 + 2x - 4)$$

$$\boxed{f(x) = -2x^4 + 6x^3 - 4x^2 - 4x + 8} \quad \underline{\text{Ans}}$$

Example 2: $3-i$ and $-2+3i$ are zeros of a polynomial function. Find the polynomial $f(x)$ of lowest degree satisfying this with $f(-1)=340$

Solution

By the Conjugate Pairs Principle, $3+i$ and $-2-3i$ are also zeros.

By the Linear Factorization Theorem, we have

$$f(x)=a\big(x-(3-i)\big)\big(x-(3+i)\big)\big(x-(-2+3i)\big)\big(x-(-2-3i)\big)$$

$$=a(x-3+i)(x-3-i)(x+2-3i)(x+2+3i)$$

$$=a\left[(x-3)^2-i^2\right]\cdot\left[(x+2)^2-9i^2\right]$$

$$=a(x^2-6x+9+1)(x^2+4x+4+9)$$

$$=a(x^2-6x+10)(x^2+4x+13)$$

$$\begin{array}{r|l} & x^2-6x+10 \\ \hline x^2 & x^4-6x^3+10x^2 \\ +4x & 4x^3-24x^2+40x \\ +13 & 13x^2-78x+130 \end{array}$$

so $f(x)=a(x^4-2x^3-x^2-38x+130)$ $\quad(*)$

Apply the side condition:

$$f(-1) = a\left[(-1)^4 - 2(-1)^3 - (-1)^2 - 38(-1) + 130\right] =$$

$$a(1 + 2 - 1 + 38 + 130) =$$

$$170a = 340$$

$$a = 2$$

Substituting back into (*):

$$f(x) = 2(x^4 - 2x^3 - x^2 - 38x + 130)$$

$$\boxed{f(x) = 2x^4 - 4x^3 - 2x^2 - 76x + 260} \quad \underline{\text{Ans}}$$