3.4E Complex Zeros III: Complex Zero Hint

A. Introduction

We consider examples of solving polynomial equations where a nonreal complex zero is given as a hint.

B. Examples

Example 1: Solve $x^4 - 6x^3 + 3x^2 + 42x - 70 = 0$ for x

Hint: $3+i$ is a zero

Solution

By the Conjugate Pairs Principle, $3-i$ is also a zero

By the Factor Theorem,

$$\left(x-(3+i)\right)\left(x-(3-i)\right) \quad \text{is a factor}$$

Thus

$$(x-3-i)(x-3+i)$$

$$[(x-3)-i]\cdot[(x-3)+i]$$

$$(x-3)^2 - i^2$$

$$(x-3)^2 + 1$$

$$x^2 - 6x + 9 + 1$$

$$x^2 - 6x + 10 \quad \text{is a factor.}$$

Divide it out using algebraic long division:

$$
\begin{array}{r l}
 & x^2 \qquad\qquad -7 \\
x^2-6x+10 & \overline{)\, x^4-6x^3+3x^2+42x-70} \\
 & \underline{-(x^4-6x^3+10x^2)} \\
 & \qquad\quad -7x^2+42x-70 \\
 & \qquad\quad \underline{-(-7x^2+42x-70)} \\
 & \qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Thus $x^4-6x^3+3x^2+42x-70=0$

$$\Downarrow$$

$$\underbrace{(x^2-6x+10)}_{\substack{\text{zeros} \\ 3+i \text{ and} \\ 3-i}}\underbrace{(x^2-7)}=0$$

$$x^2-7=0$$

$$x^2=7$$

$$x=\pm\sqrt{7}$$

Ans: $\boxed{\{-\sqrt{7}, \sqrt{7}, 3-i, 3+i\}}$

Example 2: Solve $x^4+3x^3-3x^2+21x+10=0$ for x

Hint: $1+2i$ is a zero

Solution

By the Conjugate Pairs Principle, $1-2i$ is a zero

By the Factor Theorem,

$$(x-(1+2i))(x-(1-2i)) \text{ is a factor}$$

Thus $(x-1-2i)(x-1+2i)$

$$[(x-1)-2i]\cdot[(x-1)+2i]$$

$$(x-1)^2-4i^2$$

$$(x-1)^2+4$$

$$x^2-2x+1+4$$

$$x^2-2x+5 \text{ is a factor.}$$

Divide it out using algebraic long division:

$$\begin{array}{r|l}
 & x^2+5x+2 \\
\hline
x^2-2x+5 & x^4+3x^3-3x^2+21x+10 \\
 & -(x^4-2x^3+5x^2) \\
\hline
 & 5x^3-8x^2+21x \\
 & -(5x^3-10x^2+25x) \\
\hline
 & 2x^2-4x+10 \\
 & -(2x^2-4x+10) \\
\hline
 & 0
\end{array}$$

Thus $x^4+3x^3-3x^2+21x+10=0$

$$\Downarrow$$

$$\underbrace{(x^2-2x+5)}_{\substack{\text{zeros}\\ 1+2i \text{ and}\\ 1-2i}}\underbrace{(x^2+5x+2)}=0$$

$\to x^2+5x+2=0$

QF: $x=\dfrac{-5\pm\sqrt{25-4(1)(2)}}{2}$

$x=\dfrac{-5\pm\sqrt{17}}{2}$

Ans: $\boxed{\left\{\dfrac{-5-\sqrt{17}}{2},\ \dfrac{-5+\sqrt{17}}{2},\ 1-2i,\ 1+2i\right\}}$