3.4B Irrational Zeros

A. Introduction

If a polynomial has irrational roots (like $\sqrt{7}$, .1010010001...) then the Rational Root Theorem will not help us.

EX) Let $f(x) = x^5 - x^3 - 1$

By the Rational Root Theorem, possible rational candidates are $\frac{\text{factors of } -1}{\text{factors of } 1} = \frac{\pm 1}{\pm 1} = \pm 1$

Now $f(1) = 1^5 - 1^3 - 1 = 1 - 1 - 1 = -1$

$f(-1) = (-1)^5 - (-1)^3 - 1 = -1 - (-1) - 1 = -1$

Thus f has no rational zeros.

Hence any zeros of f must be either irrational or nonreal complex.

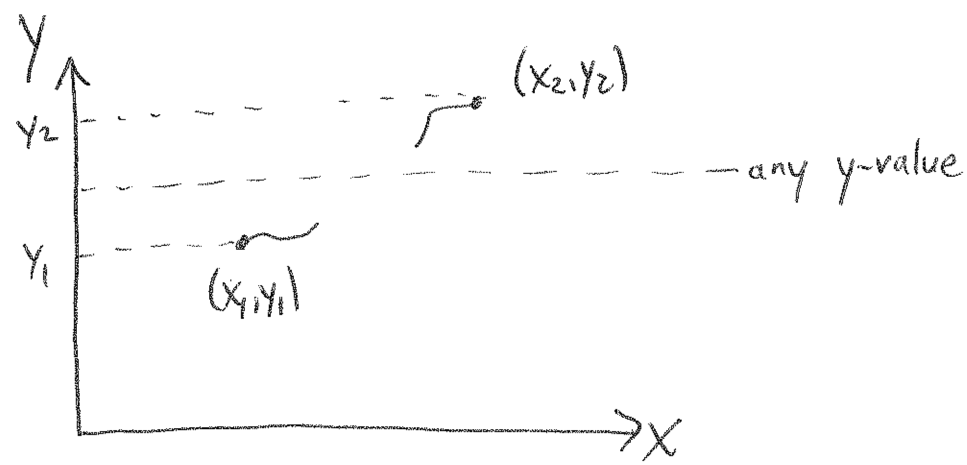
We first consider how to handle irrational zeros.

We examine some tools to help us do that...

B. Intermediate Value Theorem for Polynomials

If (x_1, y_1) and (x_2, y_2) are points that lie on the graph of a polynomial function, then

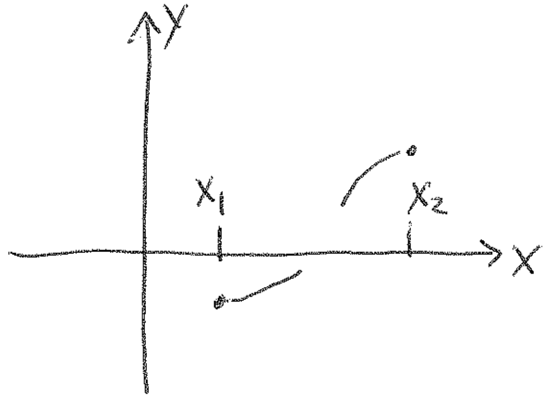
for any y-value between y_1 and y_2, we have a point (x, y) on the polynomial with x-value between x_1 and x_2.



i.e. the graph must cross the line at some point (x, y) with $x \in [x_1, x_2]$ because polynomials don't have any breaks/jumps

C. Location Principle for Polynomials

Specialize the Intermediate Value Theorem to the x-axis:



If $f(x_1) < 0$ and $f(x_2) > 0$, the polynomial function must cross the x-axis somewhere between x_1 and x_2!

Thus we have a zero somewhere between x_1 and x_2.

D. Approximating Irrational Zeros

1. By using the Location Principle, we may approximate irrational zeros as accurately as we may wish.

2. Method:
   A. Plug a sequence of integers into f
   B. When the answer switches sign (positive → negative or vice versa), we know the zero is between.
   C. Break up interval into .1 increments and repeat until sign changes.
   D. Break up interval into .01 increments, etc.
   E. Stop when you reach desired accuracy.

E. Examples

Example 1: Let $f(x) = x^5 - x^3 - 1$.

Determine an irrational zero to an accuracy of .01 for f

Solution

We already saw that f has no rational zeros.

Test a sequence of integers...

$f(0) = 0^5 - 0^3 - 1 = -1$

$f(1) = 1^5 - 1^3 - 1 = -1$

← zero must lie in here

$f(2) = 2^5 - 2^3 - 1 = 23$

Break up interval between 1 and 2 into .1 increments.

Test sequence 1, 1.1, 1.2, etc.

$f(1) = -1$

$f(1.1) \approx -.72$ (calculator)

$f(1.2) \approx -.24$ (calculator)

← zero must lie in here

$f(1.3) \approx .52$ (calculator)

Test sequence 1.2, 1.21, 1.22, etc...

$f(1.2) \approx -.24$
$f(1.21) \approx -.18$
$f(1.22) \approx -.11$
$f(1.23) \approx -.05$ ← zero lies in here
$f(1.24) \approx .03$

Since either 1.23 or 1.24 is within .01 of the zero, we are done. 1.24 is closer.

Ans: $\boxed{1.24}$

Example 2: Let $f(x) = x^4 + 2x - 2$
Determine an irrational zero to an accuracy of .01 for f

Solution

By the Rational Root Theorem, rational candidates are $\pm 1, \pm 2$

However, $f(-2) = 16 - 4 - 2 = 10$
$f(-1) = 1 - 2 - 2 = -3$
$f(1) = 1 + 2 - 2 = 1$
$f(2) = 16 + 4 - 2 = 18$

Thus there are no rational zeros, so any real zeros are irrational.

From above, we see that an irrational zero must exist between -2 and -1 for instance.

Test sequence $-2, -1.9, -1.8$, etc ...

$f(-2) = 10$

$f(-1.9) \approx 7.23$

$f(-1.8) \approx 4.90$

$f(-1.7) \approx 2.95$

$f(-1.6) \approx 1.35$

$f(-1.5) \approx .06$ $\leftarrow$ zero must lie in here

$f(-1.4) \approx -.96$

Test sequence $-1.5, -1.49, -1.48$, etc...

$f(-1.5) \approx .06$ $\leftarrow$ zero must lie in here

$f(-1.49) \approx -.05$

Thus, to an accuracy of .01, we have $\boxed{-1.49}$

Ans