3.4A Rational Zeros and Advanced Factoring Strategy

A. Introduction

We need a way to know what numbers to test, rather than just doing random guessing.

A method for coming up with a list of good guesses for rational zeros is called the Rational Root Theorem.

B. Rational Root Theorem

If $f\left(\frac{p}{q}\right)=0$, then if $f(x)=a_nx^n+a_nx^{n-1}+\ldots+a_1x+a_0=0$,

p is a factor of a_0 and q is a factor of a_n.

(Proof: See MTH310)

In other words...

$$\boxed{\text{possible rational zeros} = \frac{\text{factors of } a_0}{\text{factors of } a_n}}$$

← constant term

← leading coeff.

Note: Not every candidate will work, but if $f(x)$ has rational zeros, they must be in the list of candidates.

$f(x)$ could have irrational zeros (i.e. $\sqrt{7}$), but these can not be determined through this procedure.

C. Examples of the Rational Root Theorem

Example 1: List the possible rational zeros for f where

$$f(x) = 2x^4 - 14x^3 + 50x + 30$$

Solution

$$\text{possible rational zeros} = \frac{\text{factors of } 30}{\text{factors of } 2}$$

$$= \frac{\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30}{\pm 1, \pm 2}$$

Dividing by ± 1: $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$

Dividing by ± 2: $\pm \frac{1}{2}$, $\cancel{\pm \frac{2}{2} = \pm 1}$ repeat, $\pm \frac{3}{2}$, $\pm \frac{5}{2}$,

$\cancel{\pm \frac{6}{2} = \pm 3}$ repeat,

$\cancel{\pm \frac{10}{2} = \pm 5}$ repeat,

$\pm \frac{15}{2}$, $\cancel{\pm \frac{30}{2} = \pm 15}$ repeat

Ans: $\boxed{\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{15}{2}}$

Any rational zeros must lie in this list, so we know that 4, 7, $\frac{2}{3}$, etc. are not zeros.

Example 2: List the possible rational zeros for f where

$$f(x) = 6x^7 - 2x^3 + x^2 - 8$$

Solution

$$\text{possible rational zeros} = \frac{\text{factors of } -8}{\text{factors of } 6} = \frac{\pm 1, \pm 2, \pm 4, \pm 8}{\pm 1, \pm 2, \pm 3, \pm 6}$$

Dividing by ± 1: $\pm 1, \pm 2, \pm 4, \pm 8$

Dividing by ± 2: $\pm\frac{1}{2}, \cancel{\pm\frac{2}{2}}, \cancel{\pm\frac{4}{2}}, \cancel{\pm\frac{8}{2}}$

Dividing by ± 3: $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}$

Dividing by ± 6: $\pm\frac{1}{6}, \cancel{\pm\frac{2}{6}}, \cancel{\pm\frac{4}{6}}, \cancel{\pm\frac{8}{6}}$

Ans: $\boxed{\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}, \pm\frac{1}{6}}$

D. Advanced Factoring Strategy

1. Use the Rational Root Theorem to obtain rational zero candidates. Plug them into $f(x)$.
2. If you get $f(c)=0$ for some candidate c, then $x-c$ is a factor. Synthetically divide to get a polynomial of lower degree.
3. Repeat procedure until factored.

E. Comments

1. After obtaining the lower degree polynomial, redo the rational zero candidate list!
   Some of the original candidates dropped out!
2. It is possible that no candidates work. In this case, any real zeros must be irrational.
3. Using a calculator to graph f can sometimes give you an idea of which candidates may be likely.
   Remember zeros are x-intercepts!

F. Example

Solve $2x^4+x^3-7x^2-4x-4=0$

Solution

possible rational zeros $= \dfrac{\text{factors of } -4}{\text{factors of } 2} = \dfrac{\pm1,\pm2,\pm4}{\pm1,\pm2} = \pm1, \pm2, \pm4, \pm\frac{1}{2}$

$$f(x) = 2x^4 + x^3 - 7x^2 - 4x - 4$$

low $f(1) = 2 + 1 - 7 - 4 - 4 = -12$

$f(2) = 32 + 8 - 28 - 8 - 4 = 0$

$$\begin{array}{r|rrrrr} 2 & 2 & 1 & -7 & -4 & -4 \\ & & 4 & 10 & 6 & 4 \\ \hline & 2 & 5 & 3 & 2 & \boxed{0} \end{array}$$

So $2x^4 + x^3 - 7x^2 - 4x - 4 = (x-2)(2x^3 + 5x^2 + 3x + 2)$

possible rational zeros $= \dfrac{\text{factors of } 2}{\text{factors of } 2} = \dfrac{\pm 1, \pm 2}{\pm 1, \pm 2}$

$= \pm 1, \pm 2, \pm \frac{1}{2}$

$= -1, \pm 2, \pm \frac{1}{2}$ [already know that 1 fails!]

Let $g(x) = 2x^3 + 5x^2 + 3x + 2$

<u>Note</u>: Nothing positive works here!

Candidates: $-1, -2, -\frac{1}{2}$

$g(-1) = 2(-1)^3 + 5(-1)^2 + 3(-1) + 2 = -2 + 5 - 3 + 2 = 3$

$g(-2) = 2(-2)^3 + 5(-2)^2 + 3(-2) + 2 = -16 + 20 - 6 + 2 = 0$

$$\begin{array}{r|rrrr} -2 & 2 & 5 & 3 & 2 \\ & & -4 & -2 & -2 \\ \hline & 2 & 1 & 1 & \boxed{0} \end{array}$$

Thus $2x^4 + x^3 - 7x^2 - 4x - 4 = (x-2)(x+2)(2x^2 + x + 1)$

Now $2x^2+x+1$ is not factorable by AntiFOIL, but we can use the QF.

Thus $2x^4+x^3-7x^2-4x-4=0$ becomes

$$(x-2)(x+2)(2x^2+x+1)=0$$

By Zero Product Principle,

$$x-2=0 \quad \text{OR} \quad x+2=0 \quad \text{OR} \quad 2x^2+x+1=0$$

so $x=2, -2$ OR $x=\dfrac{-1\pm\sqrt{1^2-4(2)(1)}}{2(2)} = \dfrac{-1\pm\sqrt{-7}}{4}$

Ans: $\boxed{\left\{-2, 2, \dfrac{-1-i\sqrt{7}}{4}, \dfrac{-1+i\sqrt{7}}{4}\right\}}$

non-real complex!
we'll discuss this in
the next sections