3.3D The Remainder Theorem

A. Introduction

Consider the Division Algorithm Equation for Synthetic Division:

$$f(x) = (x-c)g(x) + r$$

Use the equation to find $f(c)$:

$$f(c) = (c-c)g(c) + r \quad = 0 \cdot g(c) + r = r$$

Thus $\boxed{f(c) = r}$

B. Remainder Theorem

Since $f(c) = r$, this says that by plugging a number into $f(x)$, we obtain the <u>remainder</u> we would get if we had done synthetic division!

This is called the <u>Remainder Theorem</u>.

Suppose $f(x) = 3x^3 - x + 5$. Then by the Remainder Theorem, the <u>remainder</u> of $f(x)$ divided by $x-1$ is the same as $f(1)$!

<u>Note</u>: $f(1) = 3(1)^3 - 1 + 5 = 3 - 1 + 5 = 7$ and

$$\begin{array}{r|rrrr} 1 & 3 & 0 & -1 & 5 \\ & & 3 & 3 & 2 \\ \hline & 3 & 3 & 2 & \boxed{7} \end{array}$$

same!

C. Examples

<u>Example 1</u>: Without doing synthetic division, what is the remainder when $2x^5 - 3x + 1$ is divided by $x-2$

<u>Solution</u>

$$f(x) = 2x^5 - 3x + 1$$

By the Remainder Theorem, we find $f(2)$:

$$f(2) = 2(2)^5 - 3(2) + 1 = 2(32) - 6 + 1 = 64 - 6 + 1$$

<u>Ans</u>: $\boxed{59}$

<u>Example 2</u>: Without doing synthetic division, what is the remainder when $2x^{101} - 5x^{50} + 3$ is divided by $x+1$?

<u>Solution</u>

$$f(x) = 2x^{101} - 5x^{50} + 3$$

By the Remainder Theorem, we find $f(-1)$:

$$f(-1) = 2(-1)^{101} - 5(-1)^{50} + 3$$

$$= 2(-1) - 5(1) + 3 = -2 - 5 + 3 = \boxed{-4}$$

<u>Ans</u>

D. Backward Examples

Sometimes the Remainder Theorem can be used "backward" to make evaluation of polynomials in difficult situations easier.

Example 1: Let $f(x) = 2x^5 - 14x^4 + x^2 - 4x + 40$

Use the Remainder Theorem (backward) to find $f(7)$ easily.

Solution

By the Remainder Theorem, $f(7)$ is the same as the remainder when dividing by $x-7$ in synthetic division:

$$\begin{array}{r|rrrrrr} 7 & 2 & -14 & 0 & 1 & -4 & 40 \\ & & 14 & 0 & 0 & 7 & 21 \\ \hline & 2 & 0 & 0 & 1 & 3 & \boxed{61} \end{array}$$

Ans: $\boxed{61}$

Example 2: Let $f(x) = 3x^4 + 32x^3 - 11x^2 + 2x + 17$

Use the Remainder Theorem (backward) to find $f(-11)$ easily

Solution

By the Remainder Theorem, $f(-11)$ is the same as the remainder when dividing by $x+11$ in synthetic division:

$$\begin{array}{r|rrrrr} -11 & 3 & 32 & -11 & 2 & 17 \\ & & -33 & 11 & 0 & -22 \\ \hline & 3 & -1 & 0 & 2 & \boxed{-5} \end{array}$$

Ans: $\boxed{5}$

E. Division Algorithm Equation with Remainder Theorem

We know:

$$f(x) = (x-c)\,g(x) + r$$

Division Algorithm Equation for Synthetic Division

and

$$f(c) = r$$

Remainder Theorem

Putting these together, we get

$$\boxed{f(x) = (x-c)\,g(x) + f(c)}$$

Division Algorithm Equation with Remainder Theorem