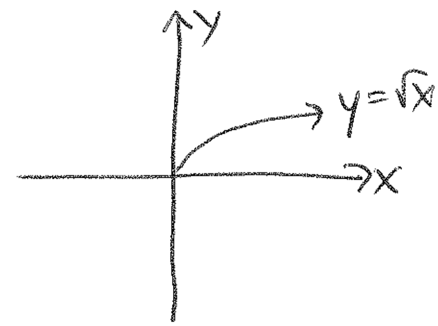
# 2.7H Proper and Improper Inverse Formulas

## A. Introduction

We observe a potential problem:

Let $f$ be the function with output formula $f(x)=\sqrt{x}$.

Then $f$ is one-to-one:




passes the HLT

(or alternately, $\sqrt{x}=\sqrt{a}$ solves to $x=a$)

Now observe what happens when we find the inverse:

$$y=\sqrt{x}$$
$$y^2=(\sqrt{x})^2$$
$$y^2=x$$
$$x=y^2$$
$$f^{-1}(y)=y^2$$
$$f^{-1}(x)=x^2$$

This seems to imply that $f$ and $g$ are inverses if $f(x)=\sqrt{x}$ and $g(x)=x^2$, which we know is false from 2.7A.

The trouble is that our candidate for the inverse, namely $f^{-1}(x)=x^2$, does not appear to define a one-to-one function "naturally".

The reason for the misunderstanding is not understanding the domain of the inverse.

Recall from Section 2.7E:

$$\operatorname{dom}(f^{-1}) = \operatorname{rng} f = [0,\infty) \quad \left[\text{by observing } f(x)=\sqrt{x}\right]$$

However, the output formula $f^{-1}(x)=x^2$ does not reflect this. Notice that $\operatorname{dom}(f^{-1})_{\text{incorrect}} = (-\infty,\infty)$.

Thus $f^{-1}(x)=x^2$ is an <u>improper</u> inverse formula.

Whenever you have an improper inverse formula, you must "correct" it by adding on a side condition ala Section 2.7F Capital Functions. The side condition is the correct domain.

Hence, $f^{-1}(x)=x^2;\ x\in[0,\infty)$ is the <u>proper</u> inverse formula.

## B. Putting It All Together

Suppose $f$ is an <u>invertible</u> function, and suppose you find $f^{-1}(x)$. If $f^{-1}(x)$ does not naturally define a one-to-one function, the formula is <u>improper</u>.

To correct the formula and make it <u>proper</u>, find $\mathrm{dom}(f^{-1})$ and add it onto $f^{-1}(x)$ as a side condition.

<u>Note:</u> Improper formulas will have $\mathrm{dom}(f^{-1})$ different from $\mathrm{dom}(f^{-1})_{\text{incorrect}}$, and this is actually the underlying problem.

## C. Example

$f$ is invertible where $f(x) = \sqrt[4]{x+3} - 5$

Find $f^{-1}(x)$ [the proper formula!]

<u>Solution</u>

$$y = \sqrt[4]{x+3} - 5$$

$$\sqrt[4]{x+3} = y+5$$

$$x+3 = (y+5)^4$$

$$x = (y+5)^4 - 3$$

$$f^{-1}(y) = (y+5)^4 - 3$$

$$f^{-1}(x) = (x+5)^4 - 3$$

not naturally one-to-one!

$$\mathrm{dom}(f^{-1}) = \mathrm{rng}\, f = [-5, \infty)$$

so

<u>Ans:</u>

$$\boxed{f^{-1}(x) = (x+5)^4 - 3;\ x \in [-5, \infty)}$$