2.7G Range Finding: Back Door Method

A. Introduction

We currently have three techniques for finding the range of a function:

1. Try to determine limitations on the function's outputs
   [Section 2.1/2.2I]

2. Graphical: Scan across the y-axis to see which ~~x~~-values get used.
   [Section 2.1/2.2N]

3. Apply HSRV Transformations to the base graph's range
   [Section 2.5F]

For some functions, these can be hard to apply.
This section introduces a fourth method which sometimes works.

The basis for it lies in the idea that $\operatorname{dom}(f^{-1}) = \operatorname{rng} f$ and in the previous ideas about capital functions.

B. Back Door Method for Finding $\operatorname{rng} f$

1. Check to see if f is invertible (i.e. one-to-one)
   If it isn't, stop here: this method won't work.
2. Find $f^{-1}(x)$ algebraically.
   If this is impossible, stop here: this method won't work
3. Check to see if $f^{-1}(x)$ defines a one-to-one function.
   If this is impossible, stop here: this method won't work.
4. Look at $f^{-1}(x)$ and find $\operatorname{dom}(f^{-1})_{\text{incorrect}}$. Then
   $\operatorname{rng} f = \operatorname{dom}(f^{-1})_{\text{incorrect}}$

C. Examples

Example 1: Use the back door method to find rng f, where $f(x) = \frac{x+1}{2x-4}$

Solution

1. Check to see if f is invertible:

   Formal method: $f(x) = f(a)$

$$\frac{x+1}{2x-4} = \frac{a+1}{2a-4}$$

$$(x+1)(2a-4) = (2x-4)(a+1)$$

$$2ax - 4x + 2a - 4 = 2ax + 2x - 4a - 4$$

$$-4x + 2a = 2x - 4a$$

$$-6x = -6a$$

$$x = a$$

f is invertible, so we can proceed...

2. Find $f^{-1}(x)$: $y = \frac{x+1}{2x-4}$

$$(2x-4)y = x+1$$

$$2xy - 4y = x+1$$

$$2xy - x = 4y+1$$

$$x(2y-1) = 4y+1$$

$$x = \frac{4y+1}{2y-1} \Rightarrow f^{-1}(y) = \frac{4y+1}{2y-1}$$

Thus $f^{-1}(x) = \frac{4x+1}{2x-1}$

3. Check to see if $\frac{4x+1}{2x-1}$ defines a one-to-one function:

Formal method: $f(x)=f(a)$

$$\frac{4x+1}{2x-1}=\frac{4a+1}{2a-1}$$

$$(4x+1)(2a-1)=(4a+1)(2x-1)$$

$$8ax-4x+2a-1=8ax+2x-4a-1$$

$$-4x+2a=2x-4a$$

$$-6x=-6a$$

$$x=a$$

Condition 3 is satisfied, so we can continue

4. Since $f^{-1}(x)=\frac{4x+1}{2x-1}$,

$\text{dom}(f^{-1})_{\text{incorrect}}$: $2x-1\neq 0$

$x\neq \frac{1}{2}$

So

$$\text{dom}(f^{-1})_{\text{incorrect}}=(-\infty,\tfrac{1}{2})\cup(\tfrac{1}{2},\infty)$$

Now $\text{rng}\, f=\text{dom}(f^{-1})_{\text{incorrect}}$, so

<u>Ans:</u> $\boxed{\text{rng}\, f=(-\infty,\tfrac{1}{2})\cup(\tfrac{1}{2},\infty)}$

Example 2: Use the back door method to find $\operatorname{rng} f$, where $f(x) = \dfrac{5}{x^2+1}$; $x \in [0,\infty)$

Solution

1. Check to see if f is invertible:

Formal Method: $f(x) = f(a)$

$$\frac{5}{x^2+1} = \frac{5}{a^2+1}$$

$$5(a^2+1) = 5(x^2+1)$$

$$5a^2+5 = 5x^2+5$$

$$5a^2 = 5x^2$$

$$\Rightarrow x^2 = a^2$$

$$\Rightarrow x = \pm a$$

Since a specific domain has been forced for f, namely $[0,\infty)$, neither x nor a can be negative

$$\Rightarrow x = a.$$

Thus f is invertible, so we can proceed...

2. Find $f^{-1}(x)$: $y = \frac{5}{x^2+1}$

$$(x^2+1)y = 5$$

$$x^2y + y = 5$$

$$x^2y = 5 - y$$

$$x^2 = \frac{5-y}{y}$$

$$x = \pm\sqrt{\frac{5-y}{y}}$$

Since $x \in [0, \infty)$, x can not be negative

$$x = \sqrt{\frac{5-y}{y}} \longrightarrow f^{-1}(y) = \sqrt{\frac{5-y}{y}}$$

so $f^{-1}(x) = \sqrt{\frac{5-x}{x}}$

3. Check to see if $\sqrt{\frac{5-x}{x}}$ defines a one-to-one function:

Formal method: $\sqrt{\frac{5-x}{x}} = \sqrt{\frac{5-a}{a}}$

$$\frac{5-x}{x} = \frac{5-a}{a}$$

$$(5-x)a = x(5-a)$$

$$5a - ax = 5x - ax$$

$$\Rightarrow 5a = 5x$$

$$\Rightarrow x = a$$

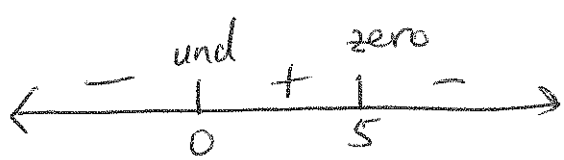
Condition 3 is satisfied

4. Now find $\text{dom}(f^{-1})_{\text{incorrect}}$ for $f^{-1}(x) = \sqrt{\frac{5-x}{x}}$

Root: $\frac{5-x}{x} \geq 0$ Denominator: $x \neq 0$

Rational Inequality

Breakpoints: 0, 5



Thus, $\text{dom}(f^{-1})_{\text{incorrect}} = (0, 5]$

Hence, $\boxed{\text{rng}(f) = (0, 5]}$ Ans

D. Comments

1. All conditions need to be checked along the way for the back door method to work.

2. Examples that the back door method would not work for:

(a) $f(x)=\sqrt{(x+3)(x-1)}$: f is not invertible
Condition 1 fails.

(b) $f(x)=x^{15}+x^5+1$: It is impossible to $f^{-1}(x)$ algebraically.
Condition 2 fails.

(c) $f(x)=\sqrt{x+4}$: $f^{-1}(x)=x^2-4$, but x^2-4 does not naturally define a one-to-one function
Condition 3 fails.