2.7F Capital Functions

A. Motivation

If f is useful, but **not** invertible (not one-to-one), we create an auxiliary function $\mathfrak{F}$ that is similar to f, but **is** invertible.

B. Capital Functions

Given f, **not** invertible . . . , we define $\mathfrak{F}$ invertible.

$\mathfrak{F}$ must have the following properties:

1. $\mathfrak{F}(x) = f(x)$ [$\mathfrak{F}$ produces the same outputs as f, so the output formula is the same]

2. $\mathfrak{F}$ is given a smaller, restricted domain of f, so that

    a. $\mathfrak{F}$ is one-to-one

    b. $\operatorname{rng} \mathfrak{F} = \operatorname{rng} f$

Any such $\mathfrak{F}$ is called a **capital function** or **principal function**

C. Method for Constructing Capital Functions

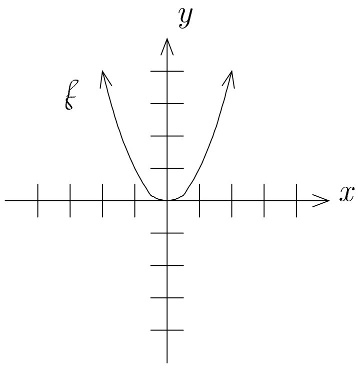
Given f, we try to determine what to cut out of $\operatorname{dom} f$ so that the result is one-to-one with the same range.

D. Examples

Example 1: f is **not** invertible, where $f(x) = x^2$. Construct a capital function $\mathfrak{I}$.

Solution

To see what is going on, let's look at the graph of f:



Note: $\mathrm{dom}\, f = (-\infty, \infty)$ and $\mathrm{rng}\, f = [0, \infty)$.

Cutting off either the left half or the right half makes the remaining part one-to-one, without changing the range.

Ans One solution is $\boxed{\mathfrak{I}(x) = x^2; x \in [0, \infty)}$

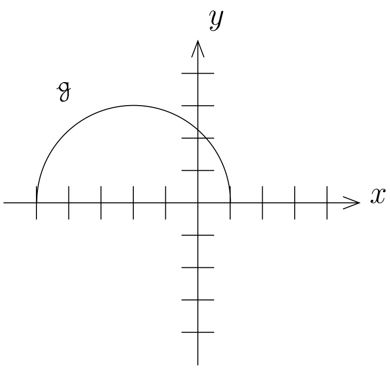
Another solution is $\boxed{\mathfrak{I}(x) = x^2; x \in (-\infty, 0]}$

Note: Each such solution can be loosely referred to as a "branch".

Example 2: g is **not** invertible, where $g(x) = \sqrt{9-(x+2)^2}$. Construct a capital function G.

Solution

To see what is going on, let's look at the graph of g:



This is the top half of a circle with center $(-2, 0)$ and radius 3.

[The full circle would be $(x+2)^2 + y^2 = 9$]

Note: $dom\ g = [-5, 1]$ and $rng\ g = [0, 3]$.

Cutting off either the left half or the right half makes the remaining part one-to-one, without changing the range.

Ans One solution is $\boxed{G(x) = \sqrt{9-(x+2)^2};\, x \in [-2, 1]}$

Another solution is $\boxed{G(x) = \sqrt{9-(x+2)^2};\, x \in [-5, -2]}$

E. Comments

1. By construction, the capital functions **are** invertible.

2. Since $\mathrm{rng}\,\mathcal{F} = \mathrm{rng}\,f$, it retains the useful information from the original function.

3. The inverse of the capital function, $\mathcal{F}^{-1}$, serves as the best approximation to an inverse of the original function one can get.

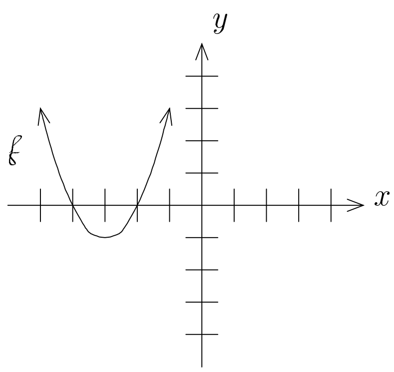
F. Inverses of the Capital Functions

Here given a function f, we examine $\mathcal{F}^{-1}$.

Example 1: Let $f(x) = (x+3)^2 - 1$. Using the left branch, determine $\mathcal{F}$ and find $\mathcal{F}^{-1}(y)$. Also, give $\mathrm{dom}(\mathcal{F}^{-1})$ and $\mathrm{rng}(\mathcal{F}^{-1})$.

Solution

Let's look at the graph of f:



Note: $\mathrm{dom}\,f = (-\infty, \infty)$ and $\mathrm{rng}\,f = [-1, \infty)$.

Using the left branch,

$$\boxed{\mathcal{F}(x) = (x+3)^2 - 1;\ x \in (-\infty, -3]}$$

Note 2: $\text{dom}\,\mathcal{F} = (-\infty, -3]$ and $\text{rng}\,\mathcal{F} = [-1, \infty)$

Now find $\mathcal{F}^{-1}$:

$$\begin{aligned} y &= (x+3)^2 - 1 \\ (x+3)^2 &= y + 1 \\ x + 3 &= \pm\sqrt{y+1} \\ x &= -3 \pm \sqrt{y+1} \end{aligned}$$

Since functions only produce one value, we need to decide which of the two solutions to take. In this case, $\text{dom}\,\mathcal{F} = (-\infty, -3]$, so we need to take the minus sign to make x smaller than -3.

Hence, $\boxed{\mathcal{F}^{-1}(y) = -3 - \sqrt{y+1}}$.

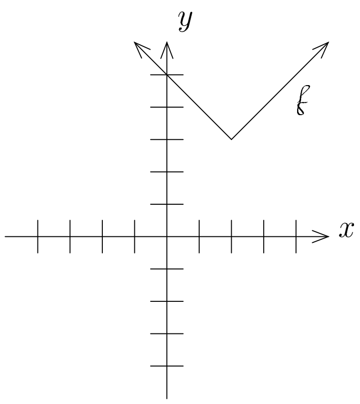
Now $\text{dom}(\mathcal{F}^{-1}) = \text{rng}\,\mathcal{F}$ and $\text{rng}(\mathcal{F}^{-1}) = \text{dom}\,\mathcal{F}$, so by Note 2, we have

$\boxed{\text{dom}(\mathcal{F}^{-1}) = [-1, \infty)}$ and $\boxed{\text{rng}(\mathcal{F}^{-1}) = (-\infty, -3]}$

Example 2: Let $f(x) = |x - 2| + 3$. Using the right branch, determine $\mathcal{F}$ and find $\mathcal{F}^{-1}(y)$. Also, give $\text{dom}(\mathcal{F}^{-1})$ and $\text{rng}(\mathcal{F}^{-1})$.

Solution

Let's look at the graph of f:



Note: $\mathit{dom}\,\mathfrak{f} = (-\infty, \infty)$ and $\mathit{rng}\,\mathfrak{f} = [3, \infty)$.

Using the right branch,

$$\boxed{\mathfrak{F}(x) = |x-2| + 3;\, x \in [2, \infty)}$$

Note 2: $\mathit{dom}\,\mathfrak{F} = [2, \infty)$ and $\mathit{rng}\,\mathfrak{F} = [3, \infty)$

Now find $\mathfrak{F}^{-1}$:

$$\begin{aligned} y &= |x-2| + 3 \\ |x-2| &= y - 3 \end{aligned}$$

Then $\quad x - 2 = y - 3 \quad \text{OR} \quad x - 2 = -(y-3).$

$$x = y - 1 \quad \text{OR} \quad x - 2 = -y + 3$$

$$x = y - 1 \quad \text{OR} \quad x = -y + 5$$

Since functions only produce one value, we need to decide which of the two solutions to take. In this case, $\mathit{dom}\,\mathfrak{F} = [2, \infty)$, so we need the equation giving x-values that are 2 or bigger. Since $y \geq 3$ [range], this happens in the **first** equation.

Hence, $\boxed{\mathfrak{F}^{-1}(y) = y - 1}$.

Now $\mathscr{dom}(\Im^{-1}) = \mathscr{rng}\,\Im$ and $\mathscr{rng}(\Im^{-1}) = \mathscr{dom}\,\Im$, so by Note 2, we have

$\boxed{\mathscr{dom}(\Im^{-1}) = [3, \infty)}$ and $\boxed{\mathscr{rng}(\Im^{-1}) = [2, \infty)}$
