2.7C Inverse Functions I

A. Notation

1. The inverse function is written as f^{-1}.

2. **Beware:** Algebra of functions is different from algebra of variables.

$$x^{-1} = \frac{1}{x} \qquad \textbf{but} \qquad f^{-1}(x) \neq \frac{1}{f(x)}$$

B. Finding Inverse Functions Algebraically

1. Verify that f is one-to-one.

2. Set output $f(x) = y$ and solve, if possible, for x (the input).

   Note: $y = f(x) \;\Rightarrow\; x = f^{-1}(y)$

C. Examples of the Algebraic Method

Example 1: Find, if possible, the output formula for f^{-1} where $f(x) = 1 - 3x$

Solution

1. Check to see if f is one-to-one:

   Formal Method: Set $f(x) = f(a)$:

   $$1 - 3x = 1 - 3a \;\Rightarrow\; -3x = -3a \;\Rightarrow\; x = a$$

   Thus f is one-to-one, and has an inverse.

2. Let $y = 1 - 3x$ and solve for x:

   $$3x + y = 1 \;\Rightarrow\; 3x = 1 - y \;\Rightarrow\; x = \tfrac{1-y}{3}$$

Ans Now $x = f^{-1}(y)$, so $\boxed{f^{-1}(y) = \tfrac{1-y}{3}}$

Note: For any output y, this formula gives back the original input x.

Example 2: Find, if possible, the output formula for f^{-1} where $f(x) = \frac{3x+4}{2x-3}$

Solution

1. Check to see if f is one-to-one:

   Formal Method: Set $f(x) = f(a)$:

   $$\frac{3x+4}{2x-3} = \frac{3a+4}{2a-3}$$

   LCD=$(2x-3)(2a-3)$ and $x \neq \frac{3}{2}$, then

   $(2x-3)(2a-3)\left[\frac{3x+4}{2x-3}\right] = (2x-3)(2a-3)\left[\frac{3a+4}{2a-3}\right]$

   $(2a-3)(3x+4) = (2x-3)(3a+4)$

   $6ax + 8a - 9x - 12 = 6ax + 8x - 9a - 12 \Rightarrow -17x = -17a \Rightarrow x = a$

   Thus f is one-to-one, and has an inverse.

2. Let $y = \frac{3x+4}{2x-3}$ and solve for x:

   Note: $x \neq \frac{3}{2}$. Then:

   $$\begin{aligned} (2x-3)y &= 3x+4 \\ 2xy - 3y &= 3x+4 \\ 2xy - 3x &= 3y+4 \\ x(2y-3) &= 3y+4 \\ x &= \frac{3y+4}{2y-3} \end{aligned}$$

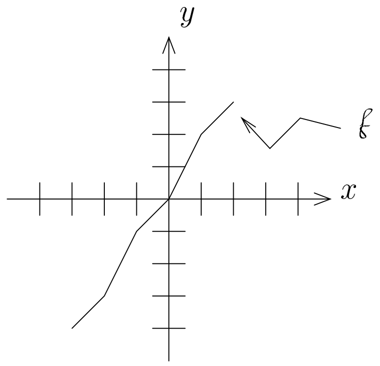
Ans Now $x = f^{-1}(y)$, so $\boxed{f^{-1}(y) = \frac{3y+4}{2y-3}}$

D. Evaluating f^{-1} on a Graph

Since $x = f^{-1}(y)$, f^{-1} takes y-values and gives back x-values. Thus, if we have the graph of f, and we want to evaluate f^{-1} at a point, we put in the y-value and take the corresponding x-value as the answer.

E. An Example

Let f be given by the following graph:



Evaluate:

a. $f^{-1}(2)$

b. $f^{-1}(-3)$

Solution

Note: f actually has an inverse, since it passes the Horizontal Line Test

a. $f^{-1}(2)$: when $y = 2$, $x = 1$, so $\boxed{f^{-1}(2) = 1}$

b. $f^{-1}(-3)$: when $y = -3$, $x = -2$, so $\boxed{f^{-1}(-3) = -2}$