2.7B One-to-One Functions

A. Motivating Question

Let's say we have a function f. We might ask the following question:

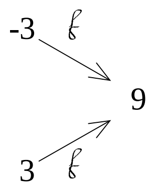
Does there **exist** a function g, so that f and g are inverses?

B. Discussion

Well, if we can find a g, at the very least we must have that $(g \circ f)(x) = g(f(x)) = x$, i.e. g undoes f (since that is one of the conditions for two functions to be inverses).

Let's consider $f(x) = x^2$ for example.

For this particular function, we have a problem. Notice what f does to -3 and 3:

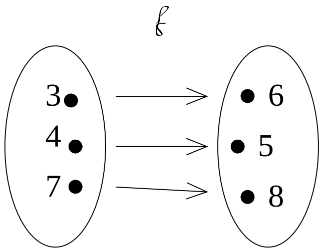


Since f sends both -3 and 3 to 9, **if** we had a g that worked, g would have to send 9 back to -3 AND 3, but functions can't do that!

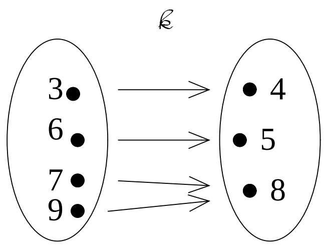
Moral: We see that for a function to have an inverse, it can **not** send two or more x's to the same number. In fact, provided the function doesn't behave badly like this, we can find an inverse (discussed later in Section 2.7C).

C. Definition of a One-to-One Function

A function is called **one-to-one** if it is **impossible** for different inputs to get sent to the same output. Alternately, we see that this means that each y can only come from one x.



one-to-one

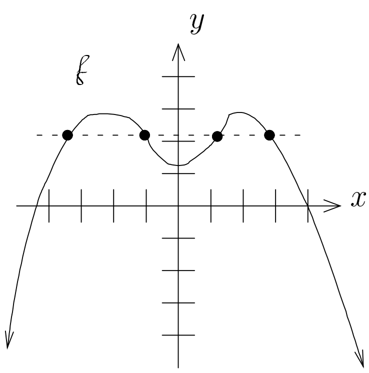


not one-to-one

D. One-to-One Tests

1. **Definition:** See if it is impossible for different x's to go to the same y.

2. **Graphical:** Horizontal Line Test
(if any horizontal line hits the graph more than once, it is not one-to-one)



not one-to-one

3. **Formal Method:**

   a. Set $f(x) = f(a)$

   b. Solve for x

   c. If $x = a$ (only), then f is one-to-one; otherwise, it is not

E. Formal Method Examples

Example 1: Determine if f is one-to-one where $f(x) = 2x + 3$

Solution

1. Set $f(x) = f(a)$: $2x + 3 = 2a + 3$

2. Solve for x: $2x = 2a \;\Rightarrow\; x = a$

Ans Since $x = a$ (only), $\boxed{f \text{ is one-to-one}}$

Example 2: Determine if g is one-to-one where $g(x) = \frac{x}{x+1}$

Solution

1. Set $g(x) = g(a)$: $\frac{x}{x+1} = \frac{a}{a+1}$

2. Solve for x: LCD=$(x + 1)(a + 1)$, and $x \neq -1$

$$(x+1)(a+1)\left[\frac{x}{x+1}\right] = (x+1)(a+1)\left[\frac{a}{a+1}\right]$$

$$x(a+1) = a(x+1) \;\Rightarrow\; ax + x = ax + a \;\Rightarrow\; x = a$$

Ans Since $x = a$ (only), $\boxed{g \text{ is one-to-one}}$

Example 3: Determine if k is one-to-one where $k(x) = \frac{x^2+4}{x^2-3}$

Solution

1. Set $k(x) = k(a)$: $\frac{x^2+4}{x^2-3} = \frac{a^2+4}{a^2-3}$

2. Solve for x: LCD=$(x^2-3)(a^2-3)$, and $x \neq \sqrt{-3}, \sqrt{3}$

$$(x^2-3)(a^2-3)\left[\frac{x^2+4}{x^2-3}\right] = (x^2-3)(a^2-3)\left[\frac{a^2+4}{a^2-3}\right]$$

$$(a^2-3)(x^2+4) = (x^2-3)(a^2+4)$$

$$a^2x^2 + 4a^2 - 3x^2 - 12 = a^2x^2 + 4x^2 - 3a^2 - 12$$

$$4a^2 - 3x^2 = 4x^2 - 3a^2 \;\Rightarrow\; -7x^2 = -7a^2 \;\Rightarrow\; x^2 = a^2 \;\Rightarrow\; x = \pm a$$

Ans Since $x = \pm a$, not just $x = a$, k is **not** one-to-one

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