2.1/2.2R Decomposition of Functions into Even and Odd Parts

A. Symmetric Domains

The domain of a function is symmetric if the domain contains the same values to the right of the origin as to the left.

i.e. symmetric domains: $(-\infty,\infty)$

$[-3,3]$

$(-4,4)$

$[-6,-2)\cup(-1,1)\cup(2,6]$

not symmetric: $[0,4]$

$(1,\infty)$

$[-2,1)$

$(-5,5]$

B. Decomposition

1. Statement: Any function (even/odd/neither) with a symmetric domain can be decomposed into the sum of an even function and an odd function.

2. Formulas:

   a. $f_{even}(x)=\frac{1}{2}[f(x)+f(-x)]$

   b. $f_{odd}(x)=\frac{1}{2}[f(x)-f(-x)]$

3. Comments

a. Note, by definition, f_{even} is an even function and f_{odd} is an odd function.

b. Note, also, that $f_{even}(x) + f_{odd}(x) = f(x)$

C. Examples

Example 1: Decompose f into even and odd parts where $f(x) = (x-1)^2$, if possible.

Solution

dom $f = (-\infty, \infty)$ symmetric $\checkmark$

Now use the formulas:

$$f_{even}(x) = \frac{1}{2}[f(x) + f(-x)]$$
$$= \frac{1}{2}[(x-1)^2 + (-x-1)^2]$$
$$= \frac{1}{2}[x^2 - 2x + 1 + x^2 + 2x + 1]$$
$$= \frac{1}{2}[2x^2 + 2] = x^2 + 1$$

$$f_{odd}(x) = \frac{1}{2}[f(x) - f(-x)]$$
$$= \frac{1}{2}[(x-1)^2 - (-x-1)^2]$$
$$= \frac{1}{2}[x^2 - 2x + 1 - (x^2 + 2x + 1)]$$
$$= \frac{1}{2}(-4x) = -2x$$

Ans: $\boxed{\begin{aligned} &f_{even}(x) = x^2+1 \\ &f_{odd}(x) = -2x \end{aligned}}$

Note: $f_{even}(x) + f_{odd}(x) = x^2+1-2x = (x-1)^2 = f(x)$

Example 2: Decompose g into even and odd parts where $g(x) = x^3$

Solution

$\text{dom } g = (-\infty, \infty)$ symmetric ✓

Use formulas:

$$\begin{aligned} g_{even}(x) &= \frac{1}{2}\left[g(x) + g(-x)\right] \\ &= \frac{1}{2}\left[x^3 + (-x)^3\right] \\ &= \frac{1}{2}(x^3 - x^3) = 0 \end{aligned}$$

$$\begin{aligned} g_{odd}(x) &= \frac{1}{2}\left[g(x) - g(-x)\right] \\ &= \frac{1}{2}\left[x^3 - (-x)^3\right] \\ &= \frac{1}{2}\left[x^3 + x^3\right] = \frac{1}{2}(2x^3) = x^3 \end{aligned}$$

Ans: $\boxed{\begin{aligned} &g_{even}(x) = 0 \\ &g_{odd}(x) = x^3 \end{aligned}}$ ← No surprise, really g [redacted] was already odd.