2.1/2.2B Equations and Functions

A. Introduction

Sometimes a relation is given by an equation, where the list of ordered pairs are the pairs that satisfy the equation.

Thus if we have the equation $x^2+y^2=9$

[a circle of radius 3 with center $(0,0)$ — we will discuss circles in more detail in Sec 2.8]

the relation is $\{(0,3), (0,-3), (3,0), (-3,0), (2,\sqrt{5}), (-2,\sqrt{5}), (2,-\sqrt{5}),$ etc. $\}$

An equation will usually generate a relation with infinitely many points!

We need to determine if relations like this, are functions.

To do so, we use the defining property of a function.

B. Defining Property of a Function

$\boxed{\text{Every valid input, } x, \text{ produces exactly one output } y.}$

Note: We will discuss what "valid" means in more detail later on. For now, we will use the rule that when talking about functions, we don't allow nonreal complex numbers [Reason: you can't plot them!]

C. Strategy for Function Testing

Plug in different values for x, and see how many y-value "answers" you get for each one.

If you get two (or more) y-values as an "answer" for a given x-value, it is not a function of x.

Otherwise, if you are reasonably confident after testing a few points that it is impossible to get more than one y-value "answer", then it is a function of x.

In the case that we do have a function of x, we further distinguish which type of function we have. Namely, if the original defining equation is solved for y, i.e. equation looks like "y = ", then we say that it is an explicit function; otherwise, we say it is an implicit function.

D. Warning About Book's Approach

The above method differs from the book's method. The book's method is, in fact, wrong.

[I can provide examples where the book's method gives the wrong answer!]

E. Examples

Determine if the following equations define functions of x. If it is a function of x, state whether it is explicit or implicit.

Example 1: $y = 3x^2 + 1$

Solution

Test x-values:

$x = 0 \Rightarrow y = 3(0)^2 + 1 = 3$ one answer

$x = 1 \Rightarrow y = 3(1)^2 + 1 = 4$ one answer

$x = -2 \Rightarrow y = 3(-2)^2 + 1 = 13$ one answer

No matter which number we choose, we can see that we always get one answer for y.

We have a function of x.

Since the original equation is solved for y, i.e. looks like $y = \text{—}$, it is an explicit function.

Ans: $\boxed{\text{YES: explicit function}}$

Example 2: $y^2 = 3x^2 + 1$

Solution: If $x=0$ is put in, say,
then $y^2 = 1 \rightarrow y = \pm 1$
Two outputs occur here!

Ans: $\boxed{\text{Not a function of } x}$

Example 3: $2x + 3y = 1$

Solution: Any value of x put in, allows you to determine one value for y. The equation is not solved for y.

Ans: $\boxed{\text{Function of } x \text{ (implicit)}}$

Example 4: $y = \sqrt{x}$

Solution: Any value of x put in, gives one value back for y
i.e. $\sqrt{4} = 2$, $\sqrt{9} = 3$

Ans: $\boxed{\text{Function of } x \text{ (explicit)}}$

Note: In Example 4, -1 is not a valid input.
Recall we don't allow nonreal complex numbers when talking about functions.

__Example 5:__ $y^2 = x$

__Solution:__ If $x=4$, say, then $y^2=4 \Rightarrow y=\pm 2$

__Ans:__ $\boxed{\text{Not a function of } x}$

__Note:__ Examples 4 and 5 differ!
$y^2=x$ and $y=\sqrt{x}$ are __not__ the same equation. In particular,
$$y^2=x \Rightarrow \sqrt{y^2}=\sqrt{x} \Rightarrow |y|=\sqrt{x} \Rightarrow y=\pm\sqrt{x}$$

__Example 6:__ $y^3=x$

__Solution:__ If $x=8$, then $y^3=8 \Rightarrow y=2$
If $x=2$, then $y^3=2 \Rightarrow y=\sqrt[3]{2}$
If $x=-27$, then $y^3=-27 \Rightarrow y=-3$
etc.

__Ans:__ $\boxed{\text{Function of } x \text{ (implicit)}}$

__Note:__ $y^3=8$ has __three__ solutions if complex numbers are allowed.
However, we only allow real numbers when discussing functions.

__General Principle:__ $y^n=a$ has __one__ __real__ solution if n is __odd__ and __up to__ __two__ __real__ solutions if n is __even__
(all other solutions are nonreal complex)