1.6 C Radical Equations

A. Introduction

These are equations with radicals [roots with variables]
Most of this section is a review of MTH1825 Sec 7.5

B. Method for Solution

1. Isolate a radical (get one radical by itself on one side)
2. Eliminate the radical by raising each side to the appropriate power
3. Repeat until all radicals are eliminated, and solve.
4. You **MUST** check your solutions. Some solutions are extraneous (i.e. fake)

C. Examples

Example 1: Solve $\sqrt[3]{2x-4}-2=0$ for x

Solution

Isolate the radical: $\sqrt[3]{2x-4}=2$

Raise each side to the third power: $\left(\sqrt[3]{2x-4}\right)^3=2^3$

Thus we have:

$$2x-4=8 \Rightarrow 2x=12 \Rightarrow x=6$$

Mandatory Check:

$$\sqrt[3]{2(6)-4}-2\stackrel{?}{=}0$$

$$\sqrt[3]{8}-2\stackrel{?}{=}0 \checkmark$$

Ans: $\boxed{\{6\}}$

Example 2: Solve $5+\sqrt{2x+5}=x$ for x

Solution

We must isolate first: $\sqrt{2x+5}=x-5$

Square each side: $\left(\sqrt{2x+5}\right)^2=(x-5)^2$

Apply square formula on right: $2x+5=x^2-10x+25$

Move everything to one side:

$$x^2-12x+20=0$$

Factor (using AntiFOIL):

$x^2-12x+20=0$	20 −, −
$x^2-x-11x+20=0$	11
$x^2-2x-10x+20=0$	20 ✓
$x(x-2)-10(x-2)=0$	

$$(x-2)(x-10)=0$$

$\Rightarrow x=2$ OR $x=10$

Mandatory Check: Check $x=2$: $5+\sqrt{2(2)+5}\overset{?}{=}2$

$5+\sqrt{9}\overset{?}{=}2$ X

Check $x=10$: $5+\sqrt{2(10)+5}\overset{?}{=}10$

$5+\sqrt{25}\overset{?}{=}10$ ✓

Ans: $\boxed{\{10\}}$

Example 3: Solve $\sqrt{2x+3} - \sqrt{x+2} = 2$ for x

Solution

We must isolate one radical first: $\sqrt{2x+3} = 2 + \sqrt{x+2}$

Square each side: $\left(\sqrt{2x+3}\right)^2 = \left(2 + \sqrt{x+2}\right)^2$

Apply square formula on right:

$$2x+3 = 4 + 4\sqrt{x+2} + (x+2)$$

$$2x+3 = 6 + x + 4\sqrt{x+2}$$

Isolate remaining radical:

$$4\sqrt{x+2} = x-3$$

$$\sqrt{x+2} = \frac{x-3}{4}$$

Square each side:

$$\left(\sqrt{x+2}\right)^2 = \left(\frac{x-3}{4}\right)^2$$

$$x+2 = \frac{(x-3)^2}{16}$$

Square formula:

$$x+2 = \frac{x^2-6x+9}{16}$$

Clear fractions:

$$x^2-6x+9 = 16(x+2)$$

$$x^2 - 6x + 9 = 16x + 32$$

Move everything to one side:

$$x^2 - 22x - 23 = 0$$

Factor (using AntiFOIL):

$$x^2 - 22x - 23 = 0 \quad \Big| \; -23 \quad +, -$$

$$x^2 + x - 23x - 23 = 0 \quad \Big| \; -23$$

$$x(x+1) - 23(x+1) = 0$$

$$(x+1)(x-23) = 0$$

$$\Rightarrow \quad x = -1 \quad \text{OR} \quad x = 23$$

<u>Mandatory Check</u>: Check $x = -1$:

$$\sqrt{2(-1)+3} - \sqrt{-1+2} \stackrel{?}{=} 2$$

$$\sqrt{1} - \sqrt{1} \stackrel{?}{=} 2$$

Check $x = 23$:

$$\sqrt{2(23)+3} - \sqrt{23+2} \stackrel{?}{=} 2$$

$$\sqrt{49} - \sqrt{25} \stackrel{?}{=} 2$$

<u>Ans</u>: $\boxed{\{23\}}$

Example 4: Solve $\sqrt{3x+4}+\sqrt{x+5}=\sqrt{7-2x}$ for x

Solution

Isolate a radical first: in fact, the radical on the right already is!

Square each side:

$$\left(\sqrt{3x+4}+\sqrt{x+5}\right)^2=\left(\sqrt{7-2x}\right)^2$$

Square formula:

$$(3x+4)+2\sqrt{3x+4}\sqrt{x+5}+(x+5)=7-2x$$

$$2\sqrt{3x+4}\sqrt{x+5}+4x+9=7-2x$$

Isolate another radical:

$$2\sqrt{3x+4}\sqrt{x+5}=-2-6x$$

$$\sqrt{x+5}=\frac{-2-6x}{2\sqrt{3x+4}}$$

$$\sqrt{x+5}=\frac{-2(1+3x)}{2\sqrt{3x+4}}$$

$$\sqrt{x+5}=\frac{-(1+3x)}{\sqrt{3x+4}}$$

Square each side!

$$\left(\sqrt{x+5}\right)^2 = \left[\frac{-(1+3x)}{\sqrt{3x+4}}\right]^2$$

$$x+5 = \frac{(1+3x)^2}{3x+4}$$

Square formula:

$$x+5 = \frac{1+6x+9x^2}{3x+4}$$

Clear fractions:

$$1+6x+9x^2 = (x+5)(3x+4)$$

$$9x^2+6x+1 = 3x^2+19x+20$$

Move to one side:

$$6x^2-13x-19=0$$

Factor (using AntiFOIL):

$6x^2-13x-19=0$	-114 $+,-$
$6x^2+x-14x-19=0$	-14
$6x^2+2x-15x-19=0$	-30
jump ahead	
$6x^2+6x-19x-19=0$	-114 ✓
$6x(x+1)-19(x+1)=0$	
$(x+1)(6x-19)=0$	

$\Rightarrow$ $x=-1$ OR $x=\frac{19}{6}$

Mandatory Check: Check $x=-1$: $\sqrt{3(-1)+4}+\sqrt{-1+5}\stackrel{?}{=}\sqrt{7-2(-1)}$

$$\sqrt{1}+\sqrt{4}\stackrel{?}{=}\sqrt{9}$$

$$1+2\stackrel{?}{=}3 \quad \checkmark$$

Check $x=\frac{19}{6}$:

$$\sqrt{3\left(\tfrac{19}{6}\right)+4}+\sqrt{\tfrac{19}{6}+5}\stackrel{?}{=}\sqrt{7-2\left(\tfrac{19}{6}\right)}$$

$$\sqrt{\tfrac{19}{2}+\tfrac{4}{1}}+\sqrt{\tfrac{19}{6}+\tfrac{5}{1}}\stackrel{?}{=}\sqrt{\tfrac{7}{1}-\tfrac{19}{3}}$$

$$\sqrt{\tfrac{19}{2}+\tfrac{8}{2}}+\sqrt{\tfrac{19}{6}+\tfrac{30}{6}}\stackrel{?}{=}\sqrt{\tfrac{21}{3}-\tfrac{19}{3}}$$

$$\sqrt{\tfrac{27}{2}}+\sqrt{\tfrac{49}{6}}\stackrel{?}{=}\sqrt{\tfrac{2}{3}}$$

$$\frac{\sqrt{27}}{\sqrt{2}}+\frac{\sqrt{49}}{\sqrt{6}}\stackrel{?}{=}\frac{\sqrt{2}}{\sqrt{3}}$$

$$\frac{3\sqrt{3}}{\sqrt{2}}+\frac{7}{\sqrt{6}}\stackrel{?}{=}\frac{\sqrt{2}}{\sqrt{3}}$$

$$\frac{3\sqrt{3}\cdot\sqrt{2}}{2}+\frac{7\sqrt{6}}{6}\stackrel{?}{=}\frac{\sqrt{2}\cdot\sqrt{3}}{3}$$

$$\frac{3\sqrt{6}}{2}+\frac{7\sqrt{6}}{6}\stackrel{?}{=}\frac{\sqrt{6}}{3}$$

$$\frac{9\sqrt{6}}{6}+\frac{7\sqrt{6}}{6}\stackrel{?}{=}\frac{\sqrt{6}}{3}$$

$$\frac{16\sqrt{6}}{6}\stackrel{?}{=}\frac{\sqrt{6}}{3}$$

$$\frac{8\sqrt{6}}{3}\stackrel{?}{=}\frac{\sqrt{6}}{3} \quad \times$$

Ans: $\boxed{\{-1\}}$

Example 5: Solve $\sqrt[3]{3x-3} - \sqrt[3]{x-2} = 1$ for x

Solution

Isolate a radical:

$$\sqrt[3]{3x-3} = 1 + \sqrt[3]{x-2}$$

Raise each side to 3rd power:

$$\left(\sqrt[3]{3x-3}\right)^3 = \left(1 + \sqrt[3]{x-2}\right)^3$$

$$3x-3 = \left(1 + \sqrt[3]{x-2}\right)^3 \quad \leftarrow \text{Evaluate cube!}$$

$$3x-3 = \left(1 + \sqrt[3]{x-2}\right)^2\left(1 + \sqrt[3]{x-2}\right)$$

$$3x-3 = \left[1 + 2\sqrt[3]{x-2} + \left(\sqrt[3]{x-2}\right)^2\right] \cdot \left[1 + \sqrt[3]{x-2}\right] \quad \text{square formula}$$

Multiply out the "3×2" product:

	1	$+\,2\sqrt[3]{x-2}$	$+\left(\sqrt[3]{x-2}\right)^2$
1	1	$+2\sqrt[3]{x-2}$	$+\left(\sqrt[3]{x-2}\right)^2$
$+\sqrt[3]{x-2}$	$\sqrt[3]{x-2}$	$+2\left(\sqrt[3]{x-2}\right)^2$	$+\left(\sqrt[3]{x-2}\right)^3$

$$3x-3 = 1 + 3\sqrt[3]{x-2} + 3\left(\sqrt[3]{x-2}\right)^2 + (x-2)$$

$$3x-3 = x - 1 + 3\sqrt[3]{x-2} + 3\left(\sqrt[3]{x-2}\right)^2$$

$$3\sqrt[3]{x-2}+3\left(\sqrt[3]{x-2}\right)^2=2x-2$$

Only one radical left, but can't isolate in one term!

Clever substitution:

$$\text{Let } u=\sqrt[3]{x-2} \qquad (1)$$

Then

$$3u+3u^2=2x-2 \qquad (2)$$

Need to get rid of x!

By (1), $u^3=x-2 \Rightarrow x=u^3+2 \quad (3)$

Substituting (3) into (2), we get

$$3u+3u^2=2(u^3+2)-2$$

$$3u+3u^2=2u^3+4-2$$

$$3u+3u^2=2u^3+2$$

Move everything to one side:

$$2u^3-3u^2-3u+2=0$$

Rearrange:

$$2u^3+2-3u^2-3u=0$$

Factor by grouping:

$$2(u^3+1) - 3u(u+1) = 0$$
$\underbrace{}$ sum of cubes!

$$2(u+1)(u^2-u+1) - 3u(u+1) = 0$$

$$(u+1)\left[2(u^2-u+1) - 3u\right] = 0$$

$$(u+1)(2u^2-2u+2-3u) = 0$$

$$(u+1)(\underbrace{2u^2-5u+2}_{\text{AntiFOIL}}) = 0$$

$2u^2-5u+2$	④ $-1, -$
$2u^2-u-4u+2$	4 ✓
$u(2u-1)-2(2u-1)$	
$(2u-1)(u-2)$	

$$(u+1)(2u-1)(u-2) = 0$$

$\Rightarrow$ $u=-1$ OR $2u-1=0$ OR $u=2$

$u=-1$ OR $u=\frac{1}{2}$ OR $u=2$

Get back to x's: use equation (1):

$\sqrt[3]{x-2} = -1$ OR $\sqrt[3]{x-2} = \frac{1}{2}$ OR $\sqrt[3]{x-2} = 2$

Cube each side:

$\left(\sqrt[3]{x-2}\right)^3 = (-1)^3$ OR $\left(\sqrt[3]{x-2}\right)^3 = \left(\frac{1}{2}\right)^3$ OR $\left(\sqrt[3]{x-2}\right)^3 = 2^3$

$x-2=-1$ OR $x-2=\frac{1}{8}$ OR $x-2=8$

$x=1$ OR $x=2+\frac{1}{8}=\frac{17}{8}$ OR $x=10$

Mandatory Check: Check $x=1$:

$$\sqrt[3]{3(1)-3} - \sqrt[3]{1-2} \stackrel{?}{=} 1$$

$$\sqrt[3]{0} - \sqrt[3]{-1} \stackrel{?}{=} 1$$

$$0-(-1) \stackrel{?}{=} 1 \checkmark$$

Check $x=\frac{17}{8}$:

$$\sqrt[3]{3\left(\frac{17}{8}\right)-3} - \sqrt[3]{\frac{17}{8}-2} \stackrel{?}{=} 1$$

$$\sqrt[3]{\frac{51}{8}-\frac{24}{8}} - \sqrt[3]{\frac{17}{8}-\frac{16}{8}} \stackrel{?}{=} 1$$

$$\sqrt[3]{\frac{27}{8}} - \sqrt[3]{\frac{1}{8}} \stackrel{?}{=} 1$$

$$\frac{3}{2}-\frac{1}{2} \stackrel{?}{=} 1 \checkmark$$

Check $x=10$:

$$\sqrt[3]{3(10)-3} - \sqrt[3]{10-2} \stackrel{?}{=} 1$$

$$\sqrt[3]{27} - \sqrt[3]{8} \stackrel{?}{=} 1$$

$$3-2 \stackrel{?}{=} 1 \checkmark$$

Ans: $\boxed{\left\{1, \frac{17}{8}, 10\right\}}$