1.6B Clever Substitutions

A. Introduction

Some weird or complicated equations can be reduced to quadratic or polynomial equations by making a "clever substitution".

B. Method

1. Identify some object "repeating" in an equation.
2. Let $u=$ "this object"
3. Use the u equation to get rid of the original variable.
4. Solve the resulting equation in the variable u.
5. Replace u with what it stands for, and solve any remaining equations for the desired variable.

C. Examples

Example 1: Solve $\left(\frac{1}{x}\right)^2-\frac{1}{x}-2=0$ for x

Solution

Let $u=\frac{1}{x}$

Then we have $u^2-u-2=0$

Factor: $(u+1)(u-2)=0$ ← AntiFOIL, if you can't see it

Thus $u=-1$ OR $u=2$

Then we have $\frac{1}{x}=-1$ OR $\frac{1}{x}=2$

Both are rational equations with LCD $=x$ and Disallowed value: $x\neq 0$

$\Rightarrow 1=-x$ OR $1=2x$ $\Rightarrow x=-1$ OR $x=\frac{1}{2}$ Ans: $\boxed{\left\{-1,\frac{1}{2}\right\}}$

Example 2: Solve $x^4+11x^2-12=0$ for x

Solution

By being clever, we can see "x^2" repeating ...

$$(x^2)^2+11x^2-12=0$$

Let $u=x^2$

$$u^2+11u-12=0$$

Solve by factoring:

$$\begin{array}{l|l} u^2+11u-12=0 & -12 \quad +,- \\ \hline u^2+12u-u-12=0 & -12 \ \checkmark \\ u(u+12)-1(u+12)=0 & \\ (u+12)(u-1)=0 & \end{array}$$

$\Rightarrow u=-12$ OR $u=1$

Then we have:

$$x^2=-12 \quad \text{OR} \quad x^2=1$$

$\Rightarrow x=\pm\sqrt{-12}$ OR $x=\pm\sqrt{1}$

$x=\pm i\sqrt{12}$ OR $x=\pm 1$

$x=\pm 2i\sqrt{3}$ OR $x=\pm 1$

Ans: $\boxed{\{-1, 1, -2i\sqrt{3}, 2i\sqrt{3}\}}$

Note: It is possible to solve this directly by factoring without making the substitution. It was done as above for demonstration.

Example 3: Solve $\left(\frac{v}{v+1}\right)^2 + \frac{2v}{v+1} = 6$ for v

Solution

Let $u = \frac{v}{v+1}$

Then we have $u^2 + 2u = 6$

$u^2 + 2u - 6 = 0$

QF: $u = \frac{-2 \pm \sqrt{4 - 4(1)(-6)}}{2(1)} = \frac{-2 \pm \sqrt{28}}{2} = \frac{-2 \pm 2\sqrt{7}}{2}$

$u = \frac{2(-1 \pm \sqrt{7})}{2} = -1 \pm \sqrt{7}$

Need to solve for v: replace u

Thus $\frac{v}{v+1} = -1 \pm \sqrt{7}$

Rational Equation(s): LCD $= v+1$
Disallowed value: $v \neq -1$

$(v+1)\left[\frac{v}{v+1}\right] = (v+1)(-1 \pm \sqrt{7})$

$v = v(-1 \pm \sqrt{7}) + 1(-1 \pm \sqrt{7})$

Isolate v:

$v - v(-1 \pm \sqrt{7}) = -1 \pm \sqrt{7}$

Factor out v:

$v\left[1 - (-1 \pm \sqrt{7})\right] = -1 \pm \sqrt{7}$

$$v = \frac{-1 \pm \sqrt{7}}{1-(-1 \pm \sqrt{7})}$$

Simplify bottom:

$$v = \frac{-1 \pm \sqrt{7}}{1+1 \mp \sqrt{7}} = \frac{-1 \pm \sqrt{7}}{2 \mp \sqrt{7}}$$ not disallowed!

Rationalize the two solutions:

Top: $\frac{-1+\sqrt{7}}{2-\sqrt{7}}$

$$= \frac{-1+\sqrt{7}}{2-\sqrt{7}} \cdot \frac{2+\sqrt{7}}{2+\sqrt{7}} = \frac{-2+\sqrt{7}+7}{4-7} = \frac{5+\sqrt{7}}{-3}$$

$$= \frac{-(5+\sqrt{7})}{3} = \frac{-5-\sqrt{7}}{3}$$

Bottom: $\frac{-1-\sqrt{7}}{2+\sqrt{7}}$

$$= \frac{-1-\sqrt{7}}{2+\sqrt{7}} \cdot \frac{2-\sqrt{7}}{2-\sqrt{7}} = \frac{-2-\sqrt{7}+7}{4-7} = \frac{5-\sqrt{7}}{-3}$$

$$= \frac{-(5-\sqrt{7})}{3} = \frac{-5+\sqrt{7}}{3}$$

<u>Ans:</u> $\left\{\frac{-5-\sqrt{7}}{3}, \frac{-5+\sqrt{7}}{3}\right\}$

Example 4: Solve $(x^2-2)^2+x^2-8=0$ for x

Solution

Clever substitution: $u=x^2-2$ (1)

Then $u^2+x^2-8=0$ (2)

Get rid of x:

By (1), $x^2=u+2$

Substituting into (2): $u^2+(u+2)-8=0$

$u^2+u-6=0$

Factor:

$(u+3)(u-2)=0$ [use AntiFOIL, if you can't see it]

Thus $u=-3$ OR $u=2$

Get back to x using (1):

$x^2-2=-3$ OR $x^2-2=2$

$x^2=-1$ OR $x^2=4$

$x=\pm\sqrt{-1}$ OR $x=\pm 2$

Ans: $\boxed{\{-2,2,-i,i\}}$