1.4 Complex Numbers

A. Introduction

<u>Recall</u>: $\sqrt{a}$ is the positive number (or zero) so that by multiplying by itself you get a

$\sqrt{9}$ is 3, because $3 \cdot 3 = 9$

<u>Q</u>: What is $\sqrt{-9}$?

<u>Ans</u>: No real number works, because <u>any</u> number times itself can not be negative

$$3 \cdot 3 = 9 \qquad [\text{never } -9!]$$
$$(-3) \cdot (-3) = 9$$

<u>Goal</u>: Resolve the annoying problem of negatives under square roots.

B. Definition of i

As above, $\sqrt{-1}$ is not a real number. We will give it a name. We will call it i.

$\boxed{i = \sqrt{-1}}$ called the <u>imaginary unit</u>.

C. Square Roots of Negatives Solved

Idea: $\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i$, we write $\sqrt{-4} = 2i$

General Rule: $\boxed{\sqrt{-a} = i\sqrt{a}}$ $(a>0)$

Ex) Simplify $\sqrt{-24}$

Solution

$i\sqrt{24} = \boxed{2i\sqrt{6}}$ Ans

D. Complex Numbers

Adding these new numbers to "ordinary" real numbers yields numbers like $3+4i$ and $2-5i$

These are called complex numbers.

A complex number is a number of the form $a+bi$, with a and b real.

Note: If $b=0$, you get a real number.
Thus all real numbers are complex numbers, but there are "many many more" complex numbers in some sense by allowing for $b \neq 0$.

E. Real and Imaginary Parts

If $z = a+bi$, then $\mathrm{Re}\, z = a$ [the <u>real part</u> of z]
and $\mathrm{Im}\, z = b$ [the <u>imaginary part</u> of z]

Thus if $z = 3+4i$, $\mathrm{Re}\, z = 3$ and $\mathrm{Im}\, z = 4$

F. Equality of Complex Numbers

When are two complex numbers equal?

Is it possible that $3+4i = 1-2i$?

The above is not possible!

Why? If $3+4i = 1-2i$, then $6i = -2$, so $i = \frac{-2}{6}$

Then $i = \frac{-1}{3}$, a real number.

We know this is not true!

This gives us the following rule:

> Two complex numbers are equal <u>if and only if</u> their real and imaginary parts are equal.

G. Example Using the Equality Rule

Solve $(2x+3)+(y-6)i = 2+3i$ for x and y

<u>Solution</u>

Using the rule: $\begin{cases} 2x+3=2 \\ y-6=3 \end{cases} \Rightarrow \begin{cases} 2x=-1 \\ y=9 \end{cases} \Rightarrow \begin{cases} \boxed{x=\frac{-1}{2}} \\ \boxed{y=9} \end{cases}$

Ans

H. Powers of i

1. $\boxed{i^2 = -1}$ by definition that $i = \sqrt{-1}$!

2. $\boxed{i^3 = -i}$

   $\boxed{\text{Why?}}$: $i^3 = i^2 \cdot i = (-1)(i) = -i$

3. $\boxed{i^4 = 1}$

   $\boxed{\text{Why?}}$: $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

Since every time you multiply 4 i's together, you get 1, you can evaluate all higher powers of i by removing <u>all</u> multiples of i^4, then use the table above.

i.e. divide and remainder trick using number 4

Examples

<u>Example1</u>: Find i^{71}

<u>Solution</u>

$$\begin{array}{r} 17 \text{ R3} \\ 4 \overline{)\,71} \\ -68 \\ \hline 3 \end{array}$$

so $i^{71} = i^3$, which is $\boxed{-i}$ <u>by table</u>

<u>Example2</u>: Find i^{106}

<u>Solution</u>

$$\begin{array}{r} 26 \text{ R2} \\ 4 \overline{)\,106} \\ -8 \\ \hline 26 \end{array}$$

so $i^{106} = i^2 = \boxed{-1}$, by table

I. Addition/Subtraction/Multiplication of Complex Numbers

Idea: Treat i like an x, but then change i^2 to -1

Examples

Example 1: Find $(2+3i)-(4+i)$

Solution

$$2+3i-4-i = \boxed{-2+2i} \text{ Ans}$$

Example 2: Find $(2+3i)(4+i)$

Solution

By FOIL, $8+2i+12i+3i^2$

$8+14i+3i^2$ change to -1!

$8+14i+3(-1)$

Ans: $\boxed{5+14i}$

Example 3: Find $(6-5i)^2$

Solution

Recall square formula [MTH1825 Section 5.1B]:

$$\underbrace{36}_{\text{square first}} - \underbrace{60i}_{\text{twice product}} + \underbrace{25i^2}_{\text{square last}}$$

$$36-60i+25(-1) = \boxed{11-60i} \text{ Ans}$$

J. Complex Conjugates

1. The <u>complex conjugate</u> of $a+bi$ is $a-bi$

[i.e. complex conjugate of $3+4i$ is $3-4i$
complex conjugate of $1-2i$ is $1+2i$]

2. The product of two complex conjugates is <u>real</u>:

$$(a+bi)(a-bi) \xrightarrow{\text{FL shortcut}} a^2-b^2i^2 = a^2-b^2(-1)$$

so $\boxed{(a+bi)(a-bi) = a^2+b^2}$

EX) Find $(2+3i)(2-3i)$

<u>Solution</u>

Using shortcut above, $2^2+3^2 = 4+9 = \boxed{13}$

K. Dividing Complex Numbers

To divide complex numbers, <u>multiply top and bottom by the complex conjugate</u>

<u>Examples:</u>

<u>Example 1</u>: Find $\frac{3+i}{2+2i}$

<u>Solution</u>

$$\frac{3+i}{2+2i} \cdot \frac{2-2i}{2-2i} \underset{\text{shortcut Section J}}{\overset{\text{FOIL}}{=\!=\!=}} \frac{6-6i+2i-2i^2}{2^2+2^2} = \frac{6-4i-2i^2}{8}$$

Change i^2 to -1:

$$\frac{6-4i+2}{8} = \frac{8-4i}{8} = \frac{4(2-i)}{8} = \frac{2-i}{2}$$

Write the answer in standard form (i.e. $a+bi$):

$$\frac{2}{2} - \frac{1}{2}i$$

Ans: $\boxed{1-\frac{1}{2}i}$

Example 2: Find $\frac{-3-4i}{i}$

Solution

complex conjugate of $i = 0+i$ is $0-i$, i.e. $-i$

$$\frac{-3-4i}{i} \cdot \frac{-i}{-i} = \frac{3i+4i^2}{-i^2} = \frac{3i+4(-1)}{-(-1)}$$

$$= \frac{3i-4}{1}$$

Ans: $\boxed{-4+3i}$ (in standard form)

L. Simplifying Expressions with Square Roots of Negatives

The radical product rule $\sqrt{a}\sqrt{b}=\sqrt{ab}$ does not work if a and b are not positive!

To simplify correctly, you must change to "i form" first!

Examples:

Example 1: Simplify $\sqrt{-6}\left(\sqrt{-15}+\sqrt{10}\right)$

Solution

Change to "i form":

$$i\sqrt{6}\left[i\sqrt{15}+\sqrt{10}\right]$$

Now multiply: $i^2\sqrt{90}+i\sqrt{60}$

$$3i^2\sqrt{10}+2i\sqrt{15}$$

$$\boxed{-3\sqrt{10}+2i\sqrt{15}}$$ Ans

Example 2: Simplify $\dfrac{6+\sqrt{-8}}{4}$

Solution

$$\frac{6+i\sqrt{8}}{4}=\frac{6+2i\sqrt{2}}{4}=\frac{2(3+i\sqrt{2})}{4}$$

Ans: $\boxed{\dfrac{3+i\sqrt{2}}{2} \text{ or } \dfrac{3}{2}+\dfrac{\sqrt{2}}{2}i}$ (standard form)