PARABOLIC MOSER ITERATION

K S

The following is a draft of my presentation demonstrating the Moser iteration technique in a parabolic setting. The proof follows Peter Li - Geometric Analysis (Lemma 19.1), with a simplification ($f \equiv 0$) indirectly recommended by Leo Abbrescia (who also presented this in a seminar). Robert Haslhofer also posted a (2 page) document online showing the technique in a simple elliptic setting, which was helpful for my understanding. (Thank you.)

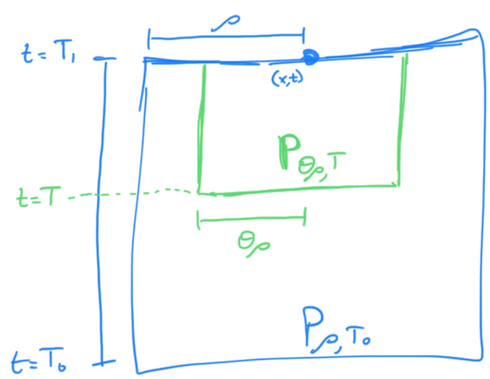
Theorem. *Suppose $u \geq 0$ satisfies*

$$u_t - \Delta u \leq fu$$

weakly on $P_{\rho,T_0} = B(\rho) \times [T_0, T_1]$, where $f = f(x,t)$ is a nonnegative function with $\sup_{[T_0,T_1]} \|f\|_{L^q(B(\rho))} < \infty$ (where $(\mu-1)/\mu < q \leq \infty$ for μ defined below). Then for any $p > 0$, and any (smaller domain scales) $\theta \in (0,1), T \in [T_0, T_1]$, there is a constant C such that

$$\|u\|_{L^\infty(P_{\theta\rho,T})} \leq C\|u\|_{L^p(P_{\rho,T_0})} \tag{1}$$

The theorem states that we can control the stronger L^∞ norm on any smaller domain by a weaker L^p norm on the bigger domain. To simplify, we will take $f \equiv 0$, since this does not hurt the exposition of the iteration.



The proof consists of two key parts. First we obtain a stepping stone version of (1). That is, we say that we can control a *slightly* stronger norm on a *slightly* smaller domain by a *slightly* weaker norm on a *slightly* bigger domain. We also keep track of how the constant depends on the changes in the domains and norm strength. The calculation of (2) may be skipped while reading.

Then we set up a sequence of shrinking domains and increasing norm strengths, which starts from the norm on the right side of (1), and which limits to the norm on the left side. We then apply the inequality from part 1 repeatedly, and in this iteration we must then show that we can control the compounding constants (that the infinite product converges).

Part 1 - Stepping stone

We will assume a Sobolev inequality in the following form. For the ρ ball in m-dimensional space, let $\mu = \frac{m}{m-2}$ (or anything greater than 2 if $m = 2$). We have a constant C_{SD} such that,

$$\fint_{B(\rho)} |\nabla \phi|^2 \geq \frac{C_{SD}}{\rho^2} \left(\fint_{B(\rho)} \phi^{2\mu} \right)^{\frac{1}{\mu}}$$

for any $\phi \in H^1_c(B(\rho))$. The important thing is that we have $\mu > 1$.

Recall that $⨍$ means the average. Note the inequality is normalized: it has the same form if we scale $B(\rho)$:

$$\frac{1}{\mathrm{Vol}(\rho)} \int_{x \in B(\rho)} |\nabla_x \phi(x)|^2 \mathrm{d}x \geq \frac{C_{SD}}{\rho^2} \left(\frac{1}{\mathrm{Vol}(\rho)} \int_{x \in B(\rho)} \phi^2(x) \mathrm{d}x \right)^{\frac{1}{\mu}}$$

$$\frac{1}{\mathrm{Vol}(\rho)} \int_{\lambda x \in B(\rho)} |\nabla_{\lambda x} \phi(\lambda x)|^2 \mathrm{d}(\lambda x) \geq \frac{C_{SD}}{\rho^2} \left(\frac{1}{\mathrm{Vol}(\rho)} \int_{\lambda x \in B(\rho)} \phi^2(\lambda x) \mathrm{d}\lambda x \right)^{\frac{1}{\mu}}$$

$$\frac{1}{\lambda^m \mathrm{Vol}(\rho/\lambda)} \int_{x \in B(\rho/\lambda)} \left| \frac{1}{\lambda} \nabla_x \phi^\lambda(x) \right|^2 \lambda^m \mathrm{d}x \geq \frac{C_{SD}}{\lambda^2 \, (\rho/\lambda)^2} \left(\frac{1}{\lambda^m \mathrm{Vol}(\rho/\lambda)} \int_{x \in B(\rho/\lambda)} (\phi^\lambda(x))^2 \lambda^m \mathrm{d}x \right)^{\frac{1}{\mu}}$$

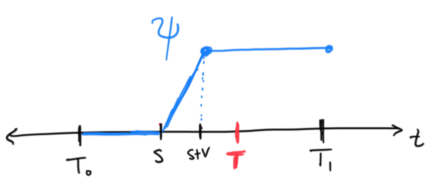
$$\frac{1}{\lambda^2} ⨍_{B(\rho/\lambda)} |\nabla \phi^\lambda|^2 \geq \frac{1}{\lambda^2} \frac{C_{SD}}{(\rho/\lambda)^2} \left(⨍_{B(\rho/\lambda)} (\phi^\lambda)^2 \right)^{\frac{1}{\mu}}$$

So we can assume that $\mathrm{Vol}(\rho) = 1$ and just use $\int$ everywhere in the calculation.

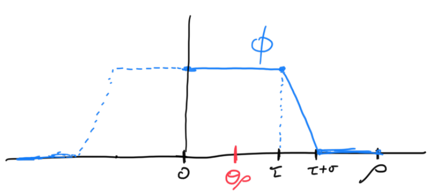
For use below we denote $C_1 := \rho^2 / C_{SD}$ and $\lambda := (2\mu - 1)/\mu > 1$.
Let $a \geq 1$. The inequality we want to prove is

$$\|u\|_{L^{2a\lambda}(P_{\tau, s+v})} \leq \left[\left(\frac{2}{\sigma^2} + \frac{2}{v} \right) C_1^{\frac{1}{\lambda}} \right]^{\frac{1}{2a}} \|u\|_{L^{2a}(P_{\tau+\sigma, s})}. \tag{2}$$

The quantities s, v, τ, σ are restricted only by $T_0 < s < s + v < T_1$ and $0 < \tau < \tau + \sigma < \rho$. In the proof, we will use the Lipschitz functions $\psi(t)$ (for time) and $\phi(r)$ (for space, radially symmetric) defined by



$$\psi(t) = \begin{cases} 0 & [T_0, s] \\ \frac{t-s}{v} & [s, s+v] \\ 1 & [s+v, T_1] \end{cases}$$



$$\phi(x) = \phi(r) = \begin{cases} 0 & B(\rho) \setminus B(\tau + \sigma) \\ \frac{\tau + \sigma - r}{\sigma} & B(\tau + \sigma) \setminus B(\tau) \\ 1 & B(\tau) \end{cases}$$

(I've marked T and $\theta\rho$ just to help visualize part 2.) So we can see that the inequality indeed controls a stronger $(2a\lambda)$ norm on a smaller domain $B(\tau) \times [s + v, T_1]$ by a weaker $(2a)$ norm on a larger domain $B(\tau + \sigma) \times [s, T_1]$. Note we use Greek letters $\sigma, \tau, \theta, \rho$ for space and Latin letters s, v, T for time.

We will now prove equation (2). We have $u_t - \Delta u \leq 0$ weakly. Using $\phi^2 u^{2a-1}$ as a test function, we get

$$\int_{B(\rho)} \langle \nabla u, \nabla(\phi^2 u^{2a-1}) \rangle + \phi^2 u_t u^{2a-1} \leq 0$$

$$= \int 2\phi u^{2a-1} \langle \nabla u, \nabla \phi \rangle + (2a-1)\phi^2 u^{2a-2} |\nabla u|^2 + \phi^2 u_t u^{2a-1} \leq 0. \tag{3}$$

Since

$$|\nabla(\phi u^a)|^2 = \left|(\nabla\phi)u^a + \phi a u^{a-1}\nabla u\right|^2 = |\nabla\phi|^2 u^{2a} + 2a\phi u^{2a-1}\langle\nabla\phi,\nabla u\rangle + \phi^2 a^2 u^{2a-2}|\nabla u|^2,$$

we can multiply (3) by a and replace the first term.

$$\int |\nabla(\phi u^a)|^2 + a(2a-1)\phi^2 u^{2a-2}|\nabla u|^2 - a^2\phi^2 u^{2a-2}|\nabla u|^2 + a\phi^2 u_t u^{2a-1} \leq \int |\nabla\phi|^2 u^{2a}$$

$$\int |\nabla(\phi u^a)|^2 + \underbrace{[a^2-a]}_{a\geq 1}\phi^2 u^{2a-2}|\nabla u|^2 + \frac{1}{2}\phi^2(u^{2a})_t \leq \int |\nabla\phi|^2 u^{2a}.$$

The second term is nonnegative so we can drop it and preserve the inequality. Also, we use the Sobolev inequality on the first term and reduce it by a half

$$\frac{C_{SD}}{\rho^2}\left(\int(\phi u^a)^{2\mu}\right)^{\frac{1}{\mu}} + \frac{1}{2}\int \phi^2(u^{2a})_t \leq \int |\nabla\phi|^2 u^{2a}$$

$$C_1^{-1}\left(\int(\phi u^a)^{2\mu}\right)^{\frac{1}{\mu}} + \int \phi^2(u^{2a})_t \leq 2\int |\nabla\phi|^2 u^{2a}. \tag{4}$$

Now multiply by ψ^2 and integrate over $[T_0, t']$. We also add a nonnegative $2\psi\psi_t\phi^2u^{2a}$ term to combine the derivative. The left side:

$$C_1^{-1}\int_{T_0}^{t'}\psi^2\left(\int_{B(\rho)}(\phi u^a)^{2\mu}\right)^{\frac{1}{\mu}}\mathrm{d}t + \int_{T_0}^{t'}\int_{B(\rho)}\psi^2\phi^2(u^{2a})_t + 2\psi\psi_t\phi^2u^{2a}\ \mathrm{d}t$$

$$= C_1^{-1}\int_{T_0}^{t'}\psi^2\left(\int_{B(\rho)}(\phi u^a)^{2\mu}\right)^{\frac{1}{\mu}}\mathrm{d}t + \int_{T_0}^{t'}\frac{\partial}{\partial t}\left(\int_{B(\rho)}\psi^2\phi^2u^{2a}\right)\mathrm{d}t$$

$$= C_1^{-1}\int_{T_0}^{t'}\psi^2\left(\int_{B(\rho)}(\phi u^a)^{2\mu}\right)^{\frac{1}{\mu}}\mathrm{d}t + \psi^2(t')\int_{B(\rho)}\phi^2u^{2a}(t') =: A + B$$

is bounded by the right side:

$$\leq \int_{T_0}^{t'}\int_{B(\rho)} 2\psi^2|\nabla\phi|^2u^{2a} + 2\psi\psi_t\phi^2u^{2a}\ \mathrm{d}t$$

$$\leq \int_{T_0}^{T_1}\int_{B(\rho)}\left(2\psi^2|\nabla\phi|^2 + 2\psi\psi_t\phi^2\right)u^{2a}\ \mathrm{d}t =: R$$

Looking at A and B individually,

$$A_{(t'=T_1)} = C_1^{-1}\int_{s+v}^{T_1}\left(\int_{B(\rho)}(\phi^2u^{2a})^{\mu}\right)^{\frac{1}{\mu}} \leq R \tag{5}$$

$$\text{and}\qquad \sup_{[s+v,T_1]}\left[\int_{B(\rho)}\phi^2u^{2a}\right] \leq R$$

We want to get a full time/space L^q-norm from the first term. Using an interpolation inequality (just a clever Hölder inequality)

$$\int g^\lambda \leq \left(\int g^\mu\right)^{\frac{1}{\mu}}\left(\int g\right)^{\frac{\mu-1}{\mu}} \qquad \text{where } \lambda = \frac{2\mu-1}{\mu}$$

on the spatial integral in (5),

$$C_1^{-1}\int_{s+v}^{T_1}\int_{B(\rho)}(\phi^2u^{2a})^\lambda \le C_1^{-1}\int_{s+v}^{T_1}\left[\left(\int_{B(\rho)}(\phi^2u^{2a})^\mu\right)^{\frac{1}{\mu}}\left(\int_{B(\rho)}\phi^2u^{2a}\right)^{\frac{\mu-1}{\mu}}\right]\mathrm{d}t$$

$$\le C_1^{-1}\left[\int_{s+v}^{T_1}\left(\int_{B(\rho)}(\phi^2u^{2a})^\mu\right)^{\frac{1}{\mu}}\mathrm{d}t\right]\left[\sup_{[s+v,T_1]}\left(\int_{B(\rho)}\phi^2u^{2a}\right)\right]^{\frac{\mu-1}{\mu}}$$

$$\le R\cdot R^{\frac{\mu-1}{\mu}} = R^\lambda$$

Dropping to the smaller domain on the left,

$$\left(\int_{s+v}^{T_1}\int_{B(\tau)}(u^{2a})^\lambda\right)^{\frac{1}{\lambda}} \le C_1^{\frac{1}{\lambda}}\int_{T_0}^{T_1}\int_{B(\rho)}(2\psi^2|\nabla\phi|^2+2\psi\psi_t\phi^2)u^{2a}.$$

We have $\psi,\phi\le 1,\ |\nabla\phi|\le\frac{1}{\sigma},\ \psi_t\le\frac{1}{v}$, and restricting to the support of ψ,ϕ,

$$\le\left(\frac{2}{\sigma^2}+\frac{2}{v}\right)C_1^{\frac{1}{\lambda}}\int_s^{T_1}\int_{B(\tau+\sigma)}u^{2a}$$

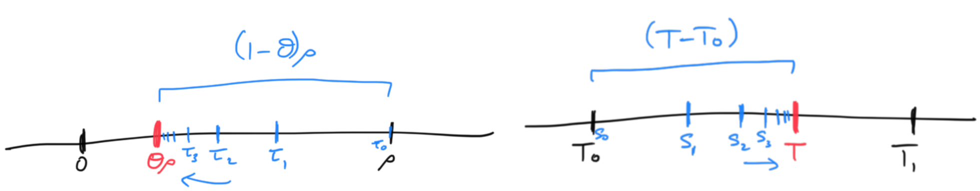
Thus we have the desired estimate

$$\left(\int_{s+v}^{T_1}\int_{B(\tau)}u^{2a\lambda}\right)^{\frac{1}{2a\lambda}}\le\left[\left(\frac{2}{\sigma^2}+\frac{2}{v}\right)C_1^{\frac{1}{\lambda}}\right]^{\frac{1}{2a}}\left(\int_s^{T_1}\int_{B(\tau+\sigma)}u^{2a}\right)^{\frac{1}{2a}}$$

$$\|u\|_{L^{2a\lambda}(s+v,\tau)}\le\left[\left(\frac{2}{\sigma^2}+\frac{2}{v}\right)C_1^{\frac{1}{\lambda}}\right]^{\frac{1}{2a}}\|u\|_{L^{2a}(s,\sigma+\tau)}.$$

Part 2 - Iteration

Choose sequences:



$$\sigma_0=\frac{(1-\theta)\rho}{2},\ \sigma_1=\frac{(1-\theta)\rho}{2^2},\ \sigma_2=\frac{(1-\theta)\rho}{2^3},\ \cdots,\ \sigma_i=\frac{(1-\theta)\rho}{2^{1+i}},\ \cdots$$

$$\tau_0=\rho,\ \tau_1=\rho-\sigma_0,\ \tau_2=\rho-\sigma_0-\sigma_1,\ \cdots,\ \tau_i=\rho-\sum_0^{i-1}\sigma_j\ \longrightarrow\theta\rho$$

$$v_0=\frac{T-T_0}{2},\ v_1=\frac{T-T_0}{2^2},\ v_2=\frac{T-T_0}{2^2},\ \cdots,\ v_i=\frac{T-T_0}{2^{1+i}},\ \cdots$$

$$s_0=T_0,\ s_1=T_0+v_0,\ s_2=T_0+v_0+v_1,\ \cdots,\ s_i=T_0+\sum_0^{i-1}v_j\longrightarrow T.$$

Also set a sequence of norm strengths:

$$a_0 = \frac{p}{2},\ a_1 = \frac{p\lambda}{2},\ a_2 = \frac{p\lambda^2}{2},\ \cdots,\ a_i = \frac{p\lambda^i}{2} \longrightarrow \infty, \text{ since } \lambda > 1.$$

Applying (2) to $s_i, v_i, \tau_{i+1}, \sigma_i, a_i$, and using $s_i + v_i = s_{i+1},\ \tau_{i+1} + \sigma_i = \tau_i,\ a_i\lambda = a_{i+1}$, we have

$$\|u\|_{L^{2a_{i+1}}(s_{i+1},\tau_{i+1})} \leq \left[\left(\frac{2}{\sigma_i^2} + \frac{2}{v_i}\right) C_1^{\frac{1}{\lambda}}\right]^{\frac{1}{2a_i}} \|u\|_{L^{2a_i}(s_i,\tau_i)}$$

$$\text{(iterating down to 0)}$$

$$\leq \left(\prod_{j=0}^{i}\left[\left(\frac{2}{\sigma_j^2} + \frac{2}{v_j}\right) C_1^{\frac{1}{\lambda}}\right]^{\frac{1}{2a_j}}\right) \|u\|_{L^p(T_0,\rho)}.$$

Meanwhile, when we shrink the domain on the left side and take the limit,

$$\|u\|_{L^{2a_{i+1}}(s_{i+1},\tau_{i+1})} \geq \|u\|_{L^{2a_{i+1}}(T,\theta\rho)} \longrightarrow \|u\|_{L^\infty(T,\theta\rho)}.$$

Thus in the limit, we have

$$\|u\|_{L^\infty(T,\theta\rho)} \leq \left(\prod_{j=0}^{\infty}\left[\left(\frac{2}{\sigma_j^2} + \frac{2}{v_j}\right) C_1^{\frac{1}{\lambda}}\right]^{\frac{1}{2a_j}}\right) \|u\|_{L^p(T_0,\rho)}.$$

All that remains is to control the constant. (This is the best part!) Writing out the forms of σ_j, v_j and a_j, the constant is

$$\prod_{j=0}^{\infty}\left[\left(\frac{2^{3+2j}}{(1-\theta)^2\rho^2} + \frac{2^{2+j}}{T-T_0}\right) C_1^{\frac{1}{\lambda}}\right]^{\frac{1}{p\lambda^j}}$$

$$\leq \prod_{j=0}^{\infty}\left(4^{\frac{1}{p}}\right)^{j\lambda^{-j}}\left[\left(\frac{2^3}{(1-\theta)^2\rho^2} + \frac{2^2}{T-T_0}\right) C_1^{\frac{1}{p\lambda}}\right]^{\lambda^{-j}}$$

$$= \left(4^{\frac{1}{p}}\right)^{\sum j\lambda^{-j}}\left[\quad \cdots \quad\right]^{\sum \lambda^{-j}}$$

Note that, since $\lambda > 1$, the sums in the exponent are finite. In fact

$$\sum_0^\infty \lambda^{-j} = \frac{1}{1-\frac{1}{\lambda}} < \infty$$

is the geometric series, and taking a derivative gives the other sum ($\sim \sum j\lambda^{-j}$). In any case the constant is a finite number C, and this proves the theorem.

$$\|u\|_{L^\infty(B(\theta\rho)\times[T,T_1])} \leq C\|u\|_{L^p(B(\rho)\times[T_0,T_1])}$$