Math 425, Homework #4 Solutions

(1) (8.9) Define a function f analytic in the plane minus the non-positive real axis and such that $f(x) = x^x$ on the positive real axis. Find $f(i)$, and $f(-i)$. Show that $f(\bar{z}) = \overline{f(z)}$ for all z in the domain.

Solution. We define an analytic branch of $\log z$ on the complement of the non-positive real axis by

$$\log z = \ln|z| + i\,\mathrm{Arg}(z)$$

where we require $\mathrm{Arg}(z)$ to take values in $(-\pi, \pi)$, or more explicitly we define

$$\mathrm{Arg}(x+iy) = \begin{cases} \cot^{-1}(x/y) & y > 0 \\ \tan^{-1}(y/x) & x > 0 \\ \cot^{-1}(x/y) - \pi & y < 0 \end{cases} \tag{1}$$

where $\tan^{-1}(t) \in (-\pi/2, \pi/2)$ and $\cot^{-1}(t) \in (0, \pi)$.

Given this analytic branch of $\log z$, we define $f(z) = e^{z \log z}$. This function is analytic one the complement of the non-positive real axis since z and $\log z$ are, and so $e^{z\log z}$ is a composition of analytic functions. Moreover, for x on the positive real axis, we have

$$\begin{aligned} f(x) &= e^{x \log x} \\ &= e^{x(\ln|x| + i\,\mathrm{Arg}(x))} \\ &= e^{x(\ln x + i0)} \\ &= e^{x \ln x} \\ &= e^{\ln x^x} = x^x. \end{aligned}$$

Next, using (1), with

$$\cot^{-1}(-t) = \pi - \cot^{-1}(t)$$

we can verify that $\mathrm{Arg}(\bar{z}) = \mathrm{Arg}(\overline{x+iy}) = \mathrm{Arg}(x-iy) = -\mathrm{Arg}(x+iy) = -\mathrm{Arg}(z)$[1], and hence the analytic branch of $\log z$ defined above satisfies

$$\begin{aligned} \log \bar{z} &= \ln|\bar{z}| + i\,\mathrm{Arg}(\bar{z}) \\ &= \ln|z| - i\,\mathrm{Arg}(z) \\ &= \overline{\ln|z| + i\,\mathrm{Arg}(z)} \\ &= \overline{\log(z)}. \end{aligned}$$

Moreover, using the definition of e^z we have that

$$\begin{aligned} e^{\bar{z}} &= e^{x-iy} \\ &= e^x(\cos(-y) + i\sin(-y)) \\ &= e^x(\cos y - i \sin y) \\ &= \overline{e^x(\cos y + i\sin y)} \\ &= \overline{e^z}. \end{aligned}$$

[1] Notice that the fact that $\mathrm{Arg}(\bar{z}) = -\mathrm{Arg}(z)$ depends crucially on our choice (1) of continuous definition of $\mathrm{Arg}(z)$ on the given domain. If we had chosen a different continuous definition of $\mathrm{Arg}(z)$ we could only say that $\mathrm{Arg}(\bar{z}) = 2\pi k(z) - \mathrm{Arg}(z)$ for some integer-valued function $k(z)$.

Consequently

$$\begin{aligned} f(\bar{z}) &= e^{\bar{z}\log\bar{z}} \\ &= e^{\bar{z}\overline{\log z}} \\ &= e^{\overline{z\log z}} \\ &= \overline{e^{z\log z}} \\ &= \overline{f(z)} \end{aligned}$$

so $f(\bar{z}) = \overline{f(z)}$ as claimed.

Finally, we compute

$$\begin{aligned} f(i) &= e^{i\log i} \\ &= e^{i(\ln|i| + i\operatorname{Arg}(i))} \\ &= e^{i(\ln 1 + i\pi/2)} \\ &= e^{-\pi/2} \end{aligned}$$

and using $f(\bar{z}) = \overline{f(z)}$, we have that

$$f(-i) = f(\bar{i}) = \overline{f(i)} = \overline{e^{-\pi/2}} = e^{-\pi/2}.$$

□

(2) Prove that the function $f(z) = \frac{\cos z - 1}{z^2}$ has a removable singularity at $z = 0$.

Proof. Using the power series representation of $\cos z$ we have that

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$$

so

$$\cos z - 1 = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$$

and therefore

$$\begin{aligned} \frac{\cos z - 1}{z^2} &= \frac{1}{z^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k-2} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+2)!} z^{2k}. \end{aligned}$$

Since the Laurent series of $f(z) = \frac{\cos z - 1}{z^2}$ at $z = 0$ has vanishing principal part, f has a removable singularity at $z = 0$ according to the theorem explained in class. □

(3) (9.10) Find the principal part of the Laurent expansion of

$$f(z) = \frac{1}{(z^2+1)^2}$$

about the point $z = i$.

Solution 1. Define $g(z) = \frac{1}{(z+i)^2}$. Since $g(z)$ is analytic at $z = i$, the Taylor series

$$g(z) = \sum_{k=0}^{\infty} \frac{g^{(k)}(i)}{k!}(z-i)^k$$

for g centered at i converges to g on some open neighborhood of i (actually on the disk $|z-i| < 2$). We then have that

$$\begin{aligned} f(z) = \frac{1}{(z^2+1)^2} &= \frac{1}{(z-i)^2} \cdot \frac{1}{(z+i)^2} \\ &= \frac{1}{(z-i)^2} \cdot g(z) \\ &= \frac{1}{(z-i)^2} \sum_{k=0}^{\infty} \frac{g^{(k)}(i)}{k!}(z-i)^k \\ &= \sum_{k=0}^{\infty} \frac{g^{(k)}(i)}{k!}(z-i)^{k-2} \\ &= \frac{g(i)}{(z-i)^2} + \frac{g'(i)}{z-i} + \sum_{k=0}^{\infty} \frac{g^{(k+2)}}{(k+2)!}(z-i)^k, \end{aligned}$$

so the principal part of $f(z)$ near $z = 1$ is

$$\frac{g(i)}{(z-i)^2} + \frac{g'(i)}{z-i}.$$

Computing, we have that $g(i) = 1/(2i)^2 = -1/4$ while $g'(z) = -2/(z+i)^3$ so $g'(i) = -2/(2i)^3 = -2/(-8i) = -i/4$. Thus the principal part of f near $z = i$ is

$$\frac{-1}{4}\left(\frac{1}{(z-i)^2} + \frac{i}{z-i}\right).$$

□

Solution 2. Using the geometric sum formula, we have for $|z-i| < 2$ that

$$\begin{aligned} \frac{-1}{z+i} &= \frac{-1}{2i + (z-i)} \\ &= \frac{-1}{2i} \frac{1}{1 - i(z-i)/2} \\ &= \frac{i}{2} \sum_{k=0}^{\infty} (i/2)^k (z-i)^k \\ &= \sum_{k=0}^{\infty} (i/2)^{k+1} (z-i)^k. \end{aligned}$$

Differentiating this formula, we get that

$$\begin{aligned}
\frac{1}{(z+i)^2} &= \frac{d}{dz}\left(\frac{-1}{z+i}\right) \\
&= \frac{d}{dz}\left(\sum_{k=0}^{\infty}(i/2)^{k+1}(z-i)^k\right) \\
&= \sum_{k=1}^{\infty}k(i/2)^{k+1}(z-i)^{k-1} \\
&= \sum_{k=0}^{\infty}(k+1)(i/2)^{k+2}(z-i)^k
\end{aligned}$$

Thus for $0 < |z-i| < 2$ we have that

$$\begin{aligned}
f(z) &= \frac{1}{(z-i)^2}\cdot\frac{1}{(z+i)^2} \\
&= \frac{1}{(z-i)^2}\sum_{k=0}^{\infty}(k+1)(i/2)^{k+2}(z-i)^k \\
&= \sum_{k=0}^{\infty}(k+1)(i/2)^{k+2}(z-i)^{k-2} \\
&= \sum_{k=-2}^{\infty}(k+3)(i/2)^{k+4}(z-i)^k.
\end{aligned}$$

Thus the principal part of f near $z = i$ is given by

$$\begin{aligned}
\frac{(-2+3)(i/2)^{-2+4}}{(z-i)^2} + \frac{(-1+3)(i/2)^{-1+4}}{(z-i)} &= \frac{(i/2)^2}{(z-i)^2} + \frac{2(i/2)^3}{(z-i)} \\
&= \frac{-1/4}{(z-i)^2} + \frac{-i/4}{(z-i)}.
\end{aligned}$$

□

(4) (9.12) Find the Laurent expansion of $f(z) = \frac{1}{z(z-1)(z-2)}$ (in powers of z) for

(a) $0 < |z| < 1$

(b) $1 < |z| < 2$

(c) $|z| > 2$.

Solution. We use partial fractions to rewrite the function

$$f(z) = \frac{1}{z(z-1)(z-2)} = \frac{1}{z}\left(\frac{1}{z-2} + \frac{-1}{z-1}\right) = \frac{1}{z}\left(\frac{-1}{2-z} + \frac{1}{1-z}\right). \tag{2}$$

In class we saw that

$$\frac{1}{1-z} = \begin{cases} \sum_{k=0}^{\infty} z^k & |z| < 1 \\ \sum_{k=-\infty}^{-1} -z^k & |z| > 1. \end{cases} \tag{3}$$

We thus find that

$$\begin{aligned} \frac{1}{2-z} &= \frac{1}{2}\left(\frac{1}{1-(z/2)}\right) \\ &= \begin{cases} \frac{1}{2}\sum_{k=0}^{\infty} (z/2)^k & |z/2| < 1 \\ \frac{1}{2}\sum_{k=-\infty}^{-1} -(z/2)^k & |z/2| > 1 \end{cases} \\ &= \begin{cases} \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} z^k & |z| < 2 \\ \sum_{k=-\infty}^{-1} -\frac{1}{2^{k+1}} z^k & |z| > 2 \end{cases} \end{aligned}$$

and hence

$$\frac{-1}{2-z} = \begin{cases} \sum_{k=0}^{\infty} -\frac{1}{2^{k+1}} z^k & |z| < 2 \\ \sum_{k=-\infty}^{-1} \frac{1}{2^{k+1}} z^k & |z| > 2. \end{cases} \tag{4}$$

Combining (2)–(4) we then find:

(a) for $0 < |z| < 1$, f can be expressed as

$$\begin{aligned} f(z) &= \frac{1}{z}\left(\sum_{k=0}^{\infty} -\frac{1}{2^{k+1}} z^k + \sum_{k=0}^{\infty} z^k\right) \\ &= \frac{1}{z}\left(\sum_{k=0}^{\infty} \left(1 - \frac{1}{2^{k+1}}\right) z^k\right) \\ &= \sum_{k=0}^{\infty} \left(1 - \frac{1}{2^{k+1}}\right) z^{k-1} \\ &= \sum_{k=-1}^{\infty} \left(1 - \frac{1}{2^{k+2}}\right) z^k. \end{aligned}$$

(b) for $1 < |z| < 2$, f can be expressed as

$$\begin{aligned} f(z) &= \frac{1}{z}\left(\sum_{k=0}^{\infty} -\frac{1}{2^{k+1}} z^k + \sum_{k=-\infty}^{-1} -z^k\right) \\ &= \sum_{k=0}^{\infty} -\frac{1}{2^{k+1}} z^{k-1} + \sum_{k=-\infty}^{-1} -z^{k-1} \\ &= \sum_{k=-1}^{\infty} -\frac{1}{2^{k+2}} z^k + \sum_{k=-\infty}^{-2} -z^k. \end{aligned}$$

(c) for $|z| > 2$, f can be expressed as

$$
\begin{aligned}
f(z) &= \frac{1}{z}\left(\sum_{k=-\infty}^{-1} \frac{1}{2^{k+1}} z^k + \sum_{k=-\infty}^{-1} -z^k\right) \\
&= \frac{1}{z}\left(\sum_{k=-\infty}^{-1} \left(\frac{1}{2^{k+1}} - 1\right) z^k\right) \\
&= \sum_{k=-\infty}^{-1} \left(\frac{1}{2^{k+1}} - 1\right) z^{k-1} \\
&= \sum_{k=-\infty}^{-2} \left(\frac{1}{2^{k+2}} - 1\right) z^k.
\end{aligned}
$$

□