Math 425, Homework #3 Solutions

(1) (4.11) Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be an analytic function on D with D an open, convex set. Suppose that f satisfies $|f'(z)| \leq 1$ for all $z \in D$. Show that

$$|f(b) - f(a)| \leq |b - a|$$

for any $a, b \in D$.

Proof. Let C be the straight line segment connecting a to b (more explicitly, we could define C to be the curve given by $z(t) = (1-t)a + tb$ for $t \in [0, 1]$). By the assumption that D is a convex set, we know that the curve C is contained in D so we can integrate $f'(z)$ along this curve and apply Proposition 4.12 to find

$$f(b) - f(a) = \int_C f'(z)\, dz. \tag{1}$$

Using the assumption that $|f'(z)| \leq 1$, we can apply the ML-formula to the above integral to conclude that

$$\left| \int_C f'(z)\, dz \right| \leq 1 \cdot \text{arclength}(C) = \text{arclength}(C).$$

Using that C is a straight line-segment connecting a to b (or computing from the arglength formula: $\text{arclength}(C) = \int_0^1 |z'(t)|\, dt = \int_0^1 |b-a|\, dt = |b-a|$) we find that the arclength of C is $|b-a|$, and so we can conclude that

$$\left| \int_C f'(z)\, dz \right| \leq |b - a|\,. \tag{2}$$

Combining (1) and (2) then allows us to conclude that

$$|f(b) - f(a)| \leq |b - a|$$

as claimed. □

(2) Compute the integral

$$\int_C \cos z\, dz$$

where C is the curve defined by

$$z(t) = te^{it} \quad t \in [0, 4\pi].$$

Answer. Using Proposition 4.12 from the course text with the fact that $\frac{d}{dz}(\sin z) = \cos z$, we compute

$$\begin{aligned}
\int_C \cos z\, dz &= \int_C \frac{d}{dz}(\sin z)\, dz \\
&= \sin z(4\pi) - \sin z(0) \\
&= \sin(4\pi e^{i4\pi}) - \sin 0 \\
&= \sin 4\pi = 0.
\end{aligned}$$

□

(3) Find a function $F : U \subset \mathbb{C} \to C$ satisfying $F'(z) = 1/z$ when:
   (a) U is the set $\{z \in \mathbb{C} \mid \operatorname{Re}(z) \neq 0\}$.
   (b) U is the set $\{z \in \mathbb{C} \mid \operatorname{Im}(z) \neq 0\}$.
   (c) U is the complement of the ray $R = \{z \in \mathbb{C} \mid \operatorname{Im}(z) = 0, \operatorname{Re}(z) \geq 0\}$.

(Hint: recall that the proof that analytic functions satisfy the Cauch-Riemann equations tells us that if $F(x+iy) = u(x,y) + iv(x,y)$, then $\frac{d}{dz}F(z) = u_x(z) + iv_x(z)$).

Proof. We begin with some preliminary observations that will be applicable to all parts of the problem. As discussed in the hint, the proof that (complex) differentiable functions satisfy the Cauch-Riemann equations shows that

$$\frac{d}{dz}F(z) = u_x(z) + iv_x(z)$$
$$(= v_y(z) - iu_y(z)).$$

Rewriting $1/z$ in terms of real and imaginary parts

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

we see that to find an F satsifying $F'(z) = 1/z$ we would need to find real functions u and v satisfying

$$u_x(x,y) = \frac{x}{x^2+y^2} \tag{3}$$

$$v_x(x,y) = \frac{-y}{x^2+y^2}. \tag{4}$$

Moreover, to ensure that $F(z)$ is analytic, we should also require that u and v satisfy the Cauchy-Riemann equations, which in this case amount to requiring that $v_y = \frac{x}{x^2+y^2}$ and $u_y = \frac{y}{x^2+y^2}$. If we define

$$u(x,y) = \frac{1}{2}\log(x^2+y^2)$$

then a straightforward computation using $\frac{d}{dt}(\log|t|) = 1/t$ and the chain rule shows that u satisfies (3) (and we point out that u also satisfies $u_y = \frac{y}{x^2+y^2}$). Therefore, in each part of the problem it remains to find a v satisfying (4), with u and v satisfying the Cauchy-Riemann equations.

As a final preliminary note, we observe that we will be using the functions $\tan^{-1}$ and $\cot^{-1}$ to express our answers below. To clear up any ambiguity about the use of these symbols we note that below we are using $\tan^{-1}(t)$ to denote the number $\theta \in (-\pi/2, \pi/2)$ satisfying $\tan\theta = t$, and we will be using $\cot^{-1}(t)$ to denote the number $\theta \in (0, \pi)$ satisfying $\cot\theta = t$.

(a) We are looking for a function v satisfying (4) on the set where $x = \operatorname{Re}(z) \neq 0$. Define $v(x,y) = \tan^{-1}(y/x)$, which has continuous first order partials wherever $x \neq 0$. Using the chain rule with $\frac{d}{dt}(\tan^{-1}(t)) = \frac{1}{1+t^2}$, we find that

$$\begin{aligned}\frac{\partial}{\partial x}\left(\tan^{-1}(y/x)\right) &= \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial x}(y/x) \\ &= \frac{1}{1+(y/x)^2} \cdot -y/x^2 \\ &= \frac{-y}{x^2+y^2}\end{aligned}$$

and a similar computation shows that also $v_y = \frac{x}{x^2+y^2}$. Thus is we define

$$F(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

F will be analytic on U because it has continuous partial derivatives and satisfies the Cauchy-Riemann equations (Proposition 3.2 in the course text), and $F'(z) = F_x(z) = 1/z$ on the set of points where $\operatorname{Re}(z) \neq 0$.

(b) Now we are looking for a function v satisfying (4) on the set where $y = \mathrm{Im}(z) \neq 0$. Define $v(x,y) = \cot^{-1}(x/y)$, which has continuous first order partial derivatives wherever $y \neq 0$. As above we use the chain rule with $\frac{d}{dt}(\cot^{-1}(t)) = \frac{-1}{1+t^2}$, to find that

$$\frac{\partial}{\partial x}\left(\cot^{-1}(x/y)\right) = \frac{-1}{1+(x/y)^2}\cdot 1/y = \frac{-y}{x^2+y^2}$$

and a similar computation shows that also $v_y = \frac{x}{x^2+y^2}$. Thus is we define

$$F(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\cot^{-1}(y/x)$$

then F will satisfy the Cauchy-Riemann equations on U and $F'(z) = F_x(z) = 1/z$ on the set of points where $\mathrm{Im}(z) \neq 0$.

(c) Here we will construct a function v satisfying (4) on the set U by defining

$$v(x,y) = \begin{cases} \cot^{-1}(y/x) & y > 0 \\ \tan^{-1}(x/y) + \pi & x < 0 \\ \cot^{-1}(y/x) + \pi & y < 0. \end{cases}$$

We will now show that v gives a well-defined function on the set U, i.e. we will show that in the "overlap regions" there is no ambiguity in the way that we defined v.

We first need to check that if $y > 0$ and $x < 0$ that

$$\cot^{-1}(y/x) = \tan^{-1}(x/y) + \pi. \tag{5}$$

From trigonometry we know that

$$\cot^{-1}(y/x) - \tan^{-1}(x/y) = k\pi \qquad \text{for some } k \in \mathbb{Z}$$

provided x and y are nonzero. Since $\cot^{-1}(y/x)$ and $\tan^{-1}(x/y)$ are continuous on the set where $y > 0$ and $x < 0$ it suffices to check that (5) holds at a single point in this region. Letting $y = 1$ and $x = -1$ we have that

$$\begin{aligned} \cot^{-1}(y/x) &= \cot^{-1}(-1) \\ &= 3\pi/4 \\ &= -\pi/4 + \pi \\ &= \tan^{-1}(-1) + \pi \\ &= \tan^{-1}(x/y) + \pi \end{aligned}$$

so (1) holds for $y = 1$ and $x = -1$ and thus holds on the set where $y > 0$ and $x < 0$.

We next check that if $y < 0$ and $x < 0$ that

$$\cot^{-1}(y/x) + \pi = \tan^{-1}(x/y) + \pi$$

or equivalently

$$\cot^{-1}(y/x) = \tan^{-1}(x/y). \tag{6}$$

Arguing as above, it suffices to check this at a single point in the region. Let $x = y = -1$. Then

$$\cot^{-1}(y/x) = \cot^{-1}(1) = \pi/4 = \tan^{-1}(1) = \tan^{-1}(x/y)$$

so (6) holds when $x = y = -1$ and thus also on the entire region where $y < 0$ and $x < 0$. Now that we've checked that $v(x,y)$ as defined above is well-defined, it follows from the same computations as in parts (a) and (b) that

$$v_x(x,y) = \frac{-y}{x^2+y^2} \qquad \text{and} \qquad v_y(x,y) = \frac{x}{x^2+y^2}.$$

Hence, with v defined as above, the function on U defined by

$$F(x,y) = \frac{1}{2}\log(x^2+y^2) + iv(x,y)$$

has continuous partial derivatives on U, satisfies the Cauchy-Riemann equations on U, and thus is analytic on U and satisfies $F'(z) = F_x(z) = 1/z$.

□

(4) Let $a \in \mathbb{C}$ be a constant, and let R be a positive real number with $R > |a|$. Use the definition of uniform convergence (i.e. give an "ε–N" proof) to prove that the series $\sum_{k=0}^{\infty} \frac{a^k}{z^{k+1}}$ converges uniformly to $f(z) = \frac{1}{z-a}$ on the circle $|z| = R$.

Proof. Let $f_n(z) = \sum_{k=0}^{n} \frac{a^k}{z^{k+1}}$ be the n-th partial sum of the series. Using the formula $\sum_{k=0}^{n} w^k = \frac{1-w^{n+1}}{1-w}$, we can rewrite the partial sum $f_n(z)$ as

$$\begin{aligned} f_n(z) &= \sum_{k=0}^{n} \frac{a^k}{z^{k+1}} = \frac{1}{z}\sum_{k=0}^{n}\left(\frac{a}{z}\right)^k \\ &= \frac{1}{z}\left(\frac{1-(a/z)^{n+1}}{1-a/z}\right) \\ &= \frac{1-(a/z)^{n+1}}{z-a}. \end{aligned}$$

Let $\varepsilon > 0$. Since $|a|/R$ is assumed to be strictly less than one, we know that the sequence $x_n = (|a|/R)^n$ converges to 0. We can therefore find an N so that

$$\left(\frac{|a|}{R}\right)^n < (R - |a|)\varepsilon \quad \text{if } n \geq N. \tag{7}$$

Then for $n \geq N$ and $|z| = R$ we will have that

$$\begin{aligned} |f_n(z) - f(z)| &= \left|\frac{1-(a/z)^{n+1}}{z-a} - \frac{1}{z-a}\right| \\ &= \left|\frac{(a/z)^{n+1}}{z-a}\right| \\ &= \frac{1}{|z-a|}\left(\frac{|a|}{|z|}\right)^{n+1} \\ &= \frac{1}{|z-a|}\left(\frac{|a|}{R}\right)^{n+1} && \text{since } |z| = R \\ &\leq \frac{1}{||z|-|a||}\left(\frac{|a|}{R}\right)^{n+1} && \text{triangle inequality } ||z|-|a|| \leq |z-a| \\ &\leq \frac{1}{R-|a|}\left(\frac{|a|}{R}\right)^{n+1} && \text{since } |z| = R > |a| \\ &< \frac{1}{R-|a|}(R-|a|)\varepsilon = \varepsilon && \text{by (7) since } n+1 > n \geq N. \end{aligned}$$

Since $|f_n(z) - f(z)| < \varepsilon$ for $|z| = R$ and $n \geq N$, we conclude that the sequence f_n converges uniformly to f on the set $|z| = R$. □