Math 425, Homework #3
Due: March 30, 2011

Instructions:

- Write in compete sentences, organized into paragraphs. Remember you should be writing your solutions so that someone who knows neither the question nor answer could read them and understand what you are proving or computing.
- Leave plenty of room in between problems.
- Write only on the front side of each sheet of paper.
- Staple!
- Write on your assignment the names of any persons or sources consulted during its completion (other than the course text or instructor).

(1) (4.11) Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be an analytic function on D with D an open, convex set.[1] Suppose that f satisfies $|f'(z)| \leq 1$ for all $z \in D$. Show that

$$|f(b) - f(a)| \leq |b - a|$$

for any $a, b \in D$.

(2) Compute the integral

$$\int_C \cos z \, dz$$

where C is the curve defined by

$$z(t) = te^{it} \quad t \in [0, 4\pi].$$

(3) Find a function $F : U \subset \mathbb{C} \to C$ satisfying $F'(z) = 1/z$ when:
(a) U is the set $\{z \in \mathbb{C} \mid \text{Re}(z) \neq 0\}$.
(b) U is the set $\{z \in \mathbb{C} \mid \text{Im}(z) \neq 0\}$.
(c) U is the complement of the ray $R = \{z \in \mathbb{C} \mid \text{Im}(z) = 0, \text{Re}(z) \geq 0\}$.
(Hint: recall that the proof that analytic functions satisfy the Cauch-Riemann equations tells us that if $F(x + iy) = u(x, y) + iv(x, y)$, then $\frac{d}{dz}F(z) = u_x(z) + iv_x(z)$).

(4) Let $a \in \mathbb{C}$ be a constant, and let R be a positive real number with $R > |a|$. Use the definition of uniform convergence (i.e. give an "ε–N" proof) to prove that the series $\sum_{k=0}^{\infty} \frac{a^k}{z^{k+1}}$ converges uniformly to $f(z) = \frac{1}{z-a}$ on the circle $|z| = R$.

[1] A set $D \subset \mathbb{C}$ is said to be convex if for every $a, b \in D$, the straight line segment connecting a to b is contained in D.