(1) Prove that for $z, w \in \mathbb{C}$, $e^z = e^w$ if and only if there exists a $k \in \mathbb{Z}$ so that $z = w + i2\pi k$.

Proof. We first consider the special case where $w = 0$. In this case, when we use the definition of $e^{x+iy} = e^x (\cos y + i \sin y)$ we find that $e^z = e^0 = 1$ is equivalent to $z = x + iy$ satisfying the system of real equations

$$e^x \cos y = 1$$
$$e^x \sin y = 0.$$

Since $e^x$ is never zero, the second equation is equivalent to $\sin y = 0$ which implies that $y = k\pi$ for $k \in \mathbb{Z}$.

Substituting this in the first equation gives us

$$1 = e^x \cos k\pi = \begin{cases} e^x & \text{if } k \text{ is even} \\ -e^x & \text{if } k \text{ is odd.} \end{cases}$$

Since $e^x$ is always positive, the odd values of $k$ do not lead to any solutions. Meanwhile the even value of $k$ give us the equation $e^x = 1$ which implies that $x = 0$. Thus we find that $e^z = 1$ precisely when $z = 0 + ik\pi$ with $k$ even, or equivalently $z = i2k\pi$ with $k \in \mathbb{Z}$.

Now, if $z$ and $w$ are any two complex numbers, the equation $e^z = e^w$ can be multiplied by $e^{-w}$ (which is never zero) to yield

$$e^{z-w} = e^{w-w} = e^0 = 1$$

where we have used the properties of the complex exponential from problem 11 in Chapter 3 of the course text. Applying the result of the first paragraph, we see that

$$e^{z-w} = 1$$

precisely when $z - w = i2\pi k$ for some $k \in \mathbb{Z}$ which is equivalent to

$$z = w + i2\pi k \text{ for } k \in \mathbb{Z}.$$
(2) (3.19) Find all solutions to the equation 

\[ e^{(e^z)} = 1. \]

**Solution.** Since 1 can be written \( 1 = e^0 \), it follow from problem (1) that \( e^{(e^z)} = 1 = e^0 \) precisely when there is a \( k \in \mathbb{Z} \) so that 

\[ e^z = i2\pi k. \] (1)

If \( k = 0 \), this has no solutions since \( e^z \) is never 0. We consider the cases \( k > 0 \) and \( k < 0 \) separately.

If \( k > 0 \) we write \( i2\pi k \) in polar/exponential form, and find that

\[ |i2\pi k| = 2\pi |k| = 2\pi k \quad \text{since } k > 0 \]

so

\[ i2\pi k = (2\pi k) i \]
\[ = e^{\log(2\pi k)} i \]
\[ = e^{\log(2\pi k)} e^{i\pi/2} \]
\[ = e^{\log(2\pi k) + i\pi/2}. \]

Thus if \( k > 0 \) the equation (1) becomes

\[ e^z = e^{\log(2\pi k) + i\pi/2} \]

which according to problem (1) is equivalent to

\[ z = \log(2\pi k) + i(\pi/2 + 2\pi n) \]

for \( n \in \mathbb{Z} \).

Meanwhile, if \( k < 0 \) to write \( i2\pi k \) in polar/exponential form, we first compute the modulus to find that

\[ |i2\pi k| = 2\pi |k| = -2\pi k \quad \text{since } k < 0 \]

so

\[ i2\pi k = (-2\pi k) (-i) \]
\[ = e^{\log(-2\pi k)} (-i) \]
\[ = e^{\log(-2\pi k)} e^{3i\pi/2} \]
\[ = e^{\log(-2\pi k) + i3\pi/2}. \]

Thus if \( k < 0 \) the equation (1) becomes

\[ e^z = e^{\log(-2\pi k) + i3\pi/2} \]

which according to problem (1) is equivalent to

\[ z = \log(-2\pi k) + i(3\pi/2 + 2\pi n) \]

for \( n \in \mathbb{Z} \).

In summary then, the solutions to \( e^{(e^z)} = 1 \) are

\[ z = \begin{cases} 
\log(2\pi k) + i(\pi/2 + 2\pi n) & \text{for } n \in \mathbb{Z} \text{ and } k \in \mathbb{Z} \cap (0, +\infty) \\
\log(-2\pi k) + i(3\pi/2 + 2\pi n) & \text{for } n \in \mathbb{Z} \text{ and } k \in \mathbb{Z} \cap (-\infty, 0). 
\end{cases} \]

We can combine these two into one expression as

\[ z = \log(2\pi |k|) + i(\pi/2 + n\pi) \quad \text{for } n \in \mathbb{Z} \text{ and } k \in \mathbb{Z} \setminus \{0\}. \]
(3) (a) Show that for complex numbers $z, w$ that

$$\cos z = \cos w \text{ and } \sin z = \sin w$$

if and only if $z = w + 2k\pi$ for some $k \in \mathbb{Z}$.

**Proof.** We first assume that $z = w + 2k\pi$ for some $k \in \mathbb{Z}$. Using the definition $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ we then compute that

$$\cos z = \cos(w + 2\pi k) = \frac{1}{2}(e^{i(w+2\pi k)} + e^{-i(w+2\pi k)})$$

$$= \frac{1}{2}(e^{i(w)} + e^{-i(w)}) = \frac{1}{2}(e^{i w + i2\pi k} + e^{-i w - i2\pi k})$$

$$= \frac{1}{2}(e^{i w}e^{i2\pi k} + e^{-i w}e^{-i2\pi k}) = \frac{1}{2}(e^{i w} + e^{-i w}) = \cos w$$

and similarly

$$\sin z = \sin(w + 2\pi k) = \frac{1}{2i}(e^{i(w+2\pi k)} - e^{-i(w+2\pi k)})$$

$$= \frac{1}{2i}(e^{i(w)} - e^{-i(w)}) = \frac{1}{2i}(e^{i w + i2\pi k} - e^{-i w - i2\pi k})$$

$$= \frac{1}{2i}(e^{i w}e^{i2\pi k} - e^{-i w}e^{-i2\pi k}) = \frac{1}{2i}(e^{i w} - e^{-i w}) = \sin w.$$ 

Therefore, if $z = w + 2k\pi$ it follows that $\cos z = \cos w$ and $\sin z = \sin w$.

We next assume that $\cos z = \cos w$ and $\sin z = \sin w$. These two equations, according to the definitions, are equivalent to

$$\frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{iw} + e^{-iw}) \quad \text{and} \quad \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2i}(e^{iw} - e^{-iw})$$

which in turn are equivalent to

$$e^{iz} + e^{-iz} = e^{iw} + e^{-iw} \quad \text{and} \quad e^{iz} - e^{-iz} = e^{iw} - e^{-iw}.$$ 

Adding these two equations together leads to

$$e^{iz} = e^{iw}.$$ 

According to problem [1] this will be true precisely when there is a $k \in \mathbb{Z}$ so that

$$iz = iw + i2\pi k$$

which is equivalent to

$$z = w + 2\pi k.$$ 

Hence $\cos z = \cos w$ and $\sin z = \sin w$ implies that $z = w + 2\pi k$ for some $k \in \mathbb{Z}$.  \[\square\]
(b) Show that
\[ \cos(z + w) = \cos z \cos w - \sin z \sin w \]
for any \( z, w \in \mathbb{C} \).

**Proof.** Using the definition of \( \cos z \) and \( \sin z \) we have that
\[
\cos z \cos w - \sin z \sin w = \left( \frac{e^{iz} + e^{-iz}}{2} \right) \left( \frac{e^{iw} + e^{-iw}}{2} \right) - \left( \frac{e^{iz} - e^{-iz}}{2i} \right) \left( \frac{e^{iw} - e^{-iw}}{2i} \right)
\]
\[
= \frac{1}{4} [(e^{iz} + e^{-iz})(e^{iw} + e^{-iw}) - \frac{1}{i^2}(e^{iz} - e^{-iz})(e^{iw} - e^{-iw})]
\]
\[
= \frac{1}{4} [(e^{iz}e^{iw} + e^{-iz}e^{iw} + e^{iz}e^{-iw} + e^{-iz}e^{-iw})
+ (e^{iz}e^{iw} - e^{-iz}e^{iw} - e^{iz}e^{-iw} + e^{-iz}e^{-iw})]
\]
\[
= \frac{1}{4} [2e^{iz}e^{iw} + 2e^{-iz}e^{-iw}]
\]
\[
= \frac{1}{2} (e^{i(z+w)} + e^{-i(z+w)})
\]
\[
= \cos(z + w).
\]

Therefore
\[ \cos(z + w) = \cos z \cos w - \sin z \sin w. \]

\( \square \)
(4) Compute \( \int_C z^2 \bar{z} \, dz \) where \( C \) is the curve defined by
\[
z(t) = t + it^2 \quad \text{for} \quad t \in [0,1].
\]

Solution. According to the definition
\[
\int_C z^2 \bar{z} \, dz = \int_0^1 z(t)^2 \bar{z}(t)z'(t) \, dt
\]
We have that
\[
z'(t) = 1 + i2t
\]
and
\[
z(t)^2 \bar{z}(t) = (t + it^2)(t^2 + t^4) = (t^3 + t^5) + i(t^4 + t^6)
\]
so
\[
z(t)^2 \bar{z}(t)z'(t) = [(t^3 + t^5) + i(t^4 + t^6)][1 + i2t]
= (t^3 + t^5 - 2t^7) + i(t^4 + t^6 + 2t^4 + 2t^6)
= (t^3 - t^5 - 2t^7) + i(3t^4 + 3t^6).
\]
We thus find that
\[
\int_C z^2 \bar{z} \, dz = \int_0^1 z(t)^2 \bar{z}(t)z'(t) \, dt
= \int_0^1 t^3 - t^5 - 2t^7 \, dt + i \int_0^1 3t^4 + 3t^6 \, dt
= \left[ \frac{t^4}{4} - \frac{t^6}{6} - \frac{t^8}{4} \right]_{t=0}^1 + i \left[ \frac{3t^5}{5} + \frac{3t^7}{7} \right]_{t=0}^1
= \frac{1}{4} - \frac{1}{6} + \frac{1}{4} + i \left( \frac{3}{5} + \frac{3}{7} \right)
= -\frac{1}{6} + i \frac{36}{35}.
\]
\[\square\]
(5) Let $u, v : \mathbb{C} \to \mathbb{R}$ be continuous functions and let $x, y : [a, b] \to U \subset \mathbb{R}$ be differentiable functions. Define $f(z) = u(z) + iv(z)$ and define $C$ to be the curve $z(t) = x(t) + iy(t)$ for $t \in [a, b]$. Find formulas for the real and imaginary parts of $\int_C f(z) \, dz$ in terms of integrals of expressions involving $u, v, x,$ and $y$.

**Solution.** Since $z'(t) = x'(t) + iy'(t)$ we use the definition of line integral to compute

$$\int_C f(z) \, dz = \int_a^b f(z(t))z'(t) \, dt$$

$$= \int_a^b [u(x(t), y(t)) + iv(x(t), y(t))] [x'(t) + iy'(t)] \, dt$$

$$= \int_a^b u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t) + i [u(x(t), y(t))y'(t) + v(x(t), y(t))x'(t)] \, dt$$

$$= \int_a^b u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t) \, dt + i \int_0^1 u(x(t), y(t))y'(t) + v(x(t), y(t))x'(t) \, dt$$

We conclude that

$$\text{Re} \left( \int_C f(z) \, dz \right) = \int_a^b u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t) \, dt$$

and

$$\text{Im} \left( \int_C f(z) \, dz \right) = \int_a^b u(x(t), y(t))y'(t) + v(x(t), y(t))x'(t) \, dt$$

\[\square\]

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1 In multivariable calculus courses (and other contexts) the integrals on the right hand side here are sometimes written in the notation

$$\text{Re} \left( \int_C f(z) \, dz \right) = \int_C u \, dx - v \, dy$$

and

$$\text{Im} \left( \int_C f(z) \, dz \right) = \int_C u \, dy + v \, dx$$

which also would have been acceptable answers.