Math 425, Homework #2 Solutions

(1) Prove that for $z, w \in \mathbb{C}$, $e^z = e^w$ if and only if there exists a $k \in \mathbb{Z}$ so that $z = w + i2\pi k$.

Proof. We first consider the special case where $w = 0$. In this case, when we use the definition of $e^{x+iy} = e^x(\cos y + i \sin y)$ we find that $e^z = e^0 = 1$ is equivalent to $z = x + iy$ satisfying the system of real equations

$$e^x \cos y = 1$$
$$e^x \sin y = 0.$$

Since e^x is never zero, the second equation is equivalent to $\sin y = 0$ which implies that

$$y = k\pi \text{ for } k \in \mathbb{Z}.$$

Substituting this in the first equation gives us

$$1 = e^x \cos k\pi = \begin{cases} e^x & \text{if } k \text{ is even} \\ -e^x & \text{if } k \text{ is odd.} \end{cases}$$

Since e^x is always positive, the odd values of k do not lead to any solutions. Meanwhile the even value of k give us the equation $e^x = 1$ which implies that $x = 0$. Thus we find that $e^z = 1$ precisely when $z = 0 + ik\pi$ with k even, or equivalently $z = i2k\pi$ with $k \in \mathbb{Z}$.

Now, if z and w are any two complex numbers, the equation $e^z = e^w$ can be multiplied by e^{-w} (which is never zero) to yield

$$e^{z-w} = e^{w-w} = e^0 = 1$$

where we have used the properties of the complex exponential from problem 11 in Chapter 3 of the course text. Applying the result of the first paragraph, we see that

$$e^{z-w} = 1$$

precisely when $z - w = i2\pi k$ for some $k \in \mathbb{Z}$ which is equivalent to

$$z = w + i2\pi k \text{ for } k \in \mathbb{Z}.$$

□

(2) (3.19) Find all solutions to the equation

$$e^{(e^z)} = 1.$$

Solution. Since 1 can be written $1 = e^0$, it follow from problem (1) that

$$e^{(e^z)} = 1 = e^0$$

precisely when there is a $k \in \mathbb{Z}$ so that

$$e^z = i2\pi k. \tag{1}$$

If $k = 0$, this has no solutions since e^z is never 0. We consider the cases $k > 0$ and $k < 0$ separately.

If $k > 0$ we write $i2\pi k$ in polar/exponential form, and find that

$$|i2\pi k| = 2\pi |k| = 2\pi k \quad \text{since } k > 0$$

so

$$\begin{aligned} i2\pi k &= (2\pi k)\, i \\ &= e^{\log(2\pi k)} i \\ &= e^{\log(2\pi k)} e^{i\pi/2} \\ &= e^{\log(2\pi k) + i\pi/2}. \end{aligned}$$

Thus if $k > 0$ the equation (1) becomes

$$e^z = e^{\log(2\pi k) + i\pi/2}$$

which according to problem (1) is equivalent to

$$z = \log(2\pi k) + i(\pi/2 + 2\pi n)$$

for $n \in \mathbb{Z}$.

Meanwhile, if $k < 0$ to write $i2\pi k$ in polar/exponential form, we first compute the modulus to find that

$$|i2\pi k| = 2\pi |k| = -2\pi k \quad \text{since } k < 0$$

so

$$\begin{aligned} i2\pi k &= (-2\pi k)\,(-i) \\ &= e^{\log(-2\pi k)} (-i) \\ &= e^{\log(-2\pi k)} e^{i3\pi/2} \\ &= e^{\log(-2\pi k) + i3\pi/2}. \end{aligned}$$

Thus if $k < 0$ the equation (1) becomes

$$e^z = e^{\log(-2\pi k) + i3\pi/2}$$

which according to problem (1) is equivalent to

$$z = \log(-2\pi k) + i(3\pi/2 + 2\pi n)$$

for $n \in \mathbb{Z}$.

In summary then, the solutions to $e^{(e^z)} = 1$ are

$$z = \begin{cases} \log(2\pi k) + i(\pi/2 + 2\pi n) & \text{for } n \in \mathbb{Z} \text{ and } k \in \mathbb{Z} \cap (0, +\infty) \\ \log(-2\pi k) + i(3\pi/2 + 2\pi n) & \text{for } n \in \mathbb{Z} \text{ and } k \in \mathbb{Z} \cap (-\infty, 0). \end{cases}$$

We can combine these two into one expression as

$$z = \log(2\pi |k|) + i(\pi/2 + n\pi) \quad \text{for } n \in \mathbb{Z} \text{ and } k \in \mathbb{Z} \setminus \{0\}.$$

□

(3) (a) Show that for complex numbers z, w that

$$\cos z = \cos w \text{ and } \sin z = \sin w$$

if and only if $z = w + 2k\pi$ for some $k \in \mathbb{Z}$.

Proof. We first assume that $z = w + 2k\pi$ for some $k \in \mathbb{Z}$. Using the definition $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ we then compute that

$$\begin{aligned}\cos z = \cos(w + 2\pi k) &= \frac{1}{2}(e^{i(w+2\pi k)} + e^{-i(w+2\pi k)}) \\ &= \frac{1}{2}(e^{i(w+2\pi k)} + e^{-i(w+2\pi k)}) = \frac{1}{2}(e^{iw+i2\pi k} + e^{-iw-i2\pi k)}) \\ &= \frac{1}{2}(e^{iw}e^{i2\pi k} + e^{-iw}e^{-i2\pi k)}) = \frac{1}{2}(e^{iw} + e^{-iw}) = \cos w\end{aligned}$$

and similarly

$$\begin{aligned}\sin z = \sin(w + 2\pi k) &= \frac{1}{2i}(e^{i(w+2\pi k)} - e^{-i(w+2\pi k)}) \\ &= \frac{1}{2i}(e^{i(w+2\pi k)} - e^{-i(w+2\pi k)}) = \frac{1}{2i}(e^{iw+i2\pi k} - e^{-iw-i2\pi k)}) \\ &= \frac{1}{2i}(e^{iw}e^{i2\pi k} - e^{-iw}e^{-i2\pi k)}) = \frac{1}{2i}(e^{iw} - e^{-iw}) = \sin w.\end{aligned}$$

Therefore, if $z = w + 2k\pi$ it follows that $\cos z = \cos w$ and $\sin z = \sin w$.

We next assume that $\cos z = \cos w$ and $\sin z = \sin w$. These two equations, according to the definitions, are equivalent to

$$\frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{iw} + e^{-iw}) \quad \text{and} \quad \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2i}(e^{iw} - e^{-iw})$$

which in turn are equivalent to

$$e^{iz} + e^{-iz} = e^{iw} + e^{-iw} \quad \text{and} \quad e^{iz} - e^{-iz} = e^{iw} - e^{-iw}.$$

Adding these two equations together leads to

$$e^{iz} = e^{iw}.$$

According to problem (1) this will be true precisely when there is a $k \in \mathbb{Z}$ so that

$$iz = iw + i2\pi k$$

which is equivalent to

$$z = w + 2\pi k.$$

Hence $\cos z = \cos w$ and $\sin z = \sin w$ implies that $z = w + 2\pi k$ for some $k \in \mathbb{Z}$. $\square$

(b) Show that

$$\cos(z+w) = \cos z \cos w - \sin z \sin w$$

for any $z,\ w \in \mathbb{C}$.

Proof. Using the definition of $\cos z$ and $\sin z$ we have that

$$\begin{aligned}
\cos z \cos w - \sin z \sin w &= \left(\tfrac{e^{iz}+e^{-iz}}{2}\right)\left(\tfrac{e^{iw}+e^{-iw}}{2}\right) - \left(\tfrac{e^{iz}-e^{-iz}}{2i}\right)\left(\tfrac{e^{iw}-e^{-iw}}{2i}\right) \\
&= \tfrac{1}{4}[(e^{iz}+e^{-iz})(e^{iw}+e^{-iw}) - \tfrac{1}{i^2}(e^{iz}-e^{-iz})(e^{iw}-e^{-iw})] \\
&= \tfrac{1}{4}[(e^{iz}e^{iw}+e^{-iz}e^{iw}+e^{iz}e^{-iw}+e^{-iz}e^{-iw}) \\
&\quad + (e^{iz}e^{iw}-e^{-iz}e^{iw}-e^{iz}e^{-iw}x+e^{-iz}e^{-iw})] \\
&= \tfrac{1}{4}[2e^{iz}e^{iw}+2e^{-iz}e^{-iw}] \\
&= \tfrac{1}{2}(e^{i(z+w)}+e^{-i(z+w)}) \\
&= \cos(z+w).
\end{aligned}$$

Therefore

$$\cos(z+w) = \cos z \cos w - \sin z \sin w.$$

□

(4) Compute $\int_C z^2 \bar{z}\, dz$ where C is the curve defined by

$$z(t) = t + it^2 \text{ for } t \in [0, 1].$$

Solution. According to the definition

$$\int_C z^2 \bar{z}\, dz = \int_0^1 z(t)^2 \bar{z}(t) z'(t)\, dt$$

We have that

$$z'(t) = 1 + i2t$$

and

$$z(t)^2 \bar{z}(t) = z(t)\, |z(t)|^2 = (t + it^2)(t^2 + t^4) = (t^3 + t^5) + i(t^4 + t^6)$$

so

$$\begin{aligned} z(t)^2 \bar{z}(t) z'(t) &= [(t^3 + t^5) + i(t^4 + t^6)][1 + i2t] \\ &= (t^3 + t^5 - 2t^5 - 2t^7) + i(t^4 + t^6 + 2t^4 + 2t^6) \\ &= (t^3 - t^5 - 2t^7) + i(3t^4 + 3t^6). \end{aligned}$$

We thus find that

$$\begin{aligned} \int_C z^2 \bar{z}\, dz &= \int_0^1 z(t)^2 \bar{z}(t) z'(t)\, dt \\ &= \int_0^1 t^3 - t^5 - 2t^7\, dt + i \int_0^1 3t^4 + 3t^6\, dt \\ &= \left[\frac{t^4}{4} - \frac{t^6}{6} - \frac{t^8}{4}\right]_{t=0}^1 + i \left[\frac{3t^5}{5} + \frac{3t^7}{7}\right]_{t=0}^1 \\ &= \frac{1}{4} - \frac{1}{6} - \frac{1}{4} + i\left(\frac{3}{5} + \frac{3}{7}\right) \\ &= -\frac{1}{6} + i\frac{36}{35}. \end{aligned}$$

□

(5) Let u, $v : \mathbb{C} \to \mathbb{R}$ be continuous functions and let x, $y : [a, b] \to U \subset \mathbb{R}$ be differentiable functions. Define $f(z) = u(z) + iv(z)$ and define C to be the curve $z(t) = x(t) + iy(t)$ for $t \in [a, b]$. Find formulas for the real and imaginary parts of $\int_C f(z)\,dz$ in terms of integrals of expressions involving u, v, x, and y.

Solution. Since

$$z'(t) = x'(t) + iy'(t)$$

we use the definition of line integral to compute

$$\begin{aligned}
\int_C f(z)\,dz &= \int_a^b f(z(t))z'(t)\,dt \\
&= \int_a^b [u(x(t), y(t)) + iv(x(t), y(t))][x'(t) + iy'(t)]\,dt \\
&= \int_a^b u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t) + i\,[u(x(t), y(t))y'(t) + v(x(t), y(t))x'(t)]\ dt \\
&= \int_a^b u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)\,dt + i\int_0^1 u(x(t), y(t))y'(t) + v(x(t), y(t))x'(t)\,dt
\end{aligned}$$

We conclude that[1]

$$\mathrm{Re}\left(\int_C f(z)\,dz\right) = \int_a^b u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)\,dt$$

and

$$\mathrm{Im}\left(\int_C f(z)\,dz\right) = \int_a^b u(x(t), y(t))y'(t) + v(x(t), y(t))x'(t)\,dt$$

□

[1] In multivariable calculus courses (and other contexts) the integrals on the right hand side here are sometimes written in the notation

$$\mathrm{Re}\left(\int_C f(z)\,dz\right) = \int_C u\,dx - v\,dy$$

and

$$\mathrm{Im}\left(\int_C f(z)\,dz\right) = \int_C u\,dy + v\,dx$$

which also would have been acceptable answers.