Math 425, Homework #1 Solutions

(1) (1.9)

(a) Use complex algebra to show that for any four integers a, b, c, and d there are integers u and v so that
$$(a^2+b^2)(c^2+d^2) = u^2+v^2$$

Proof. If we let $z_1 = a+ib$ and $z_2 = c+id$ then we have that
$$\begin{aligned}(a^2+b^2)(c^2+d^2) &= |z_1|^2\,|z_2|^2 \\ &= z_1\bar{z}_1 z_2\bar{z}_2 \\ &= z_1z_2\overline{z_1z_2} = |z_1z_2|^2 = \mathrm{Re}(z_1z_2)^2 + \mathrm{Im}(z_1z_2)^2 \\ &= z_1\bar{z}_2(\bar{z}_1z_2) = z_1\bar{z}_2\overline{(z_1\bar{z}_2)} = |z_1\bar{z}_2|^2 = \mathrm{Re}(z_1\bar{z}_2)^2 + \mathrm{Im}(z_1\bar{z}_2)^2.\end{aligned}$$
Since $z_1z_2 = (ac-bd)+i(ad+bc)$ and $z_1\bar{z}_2 = (ac+bd)+i(bc-ad)$ appropriate values of u and v would be given by
$$u = ac-bd \text{ and } v = ad+bc$$
or
$$u = ac+bd \text{ and } v = bc-ad.$$
Note that these choices of u and v are integers because products of integers are integers and sums/differences of integers are integers. □

(b) Assume that the integers a, b, c, and d are all nonzero and that $a^2 \neq b^2$. Show that we can find integers u and v satisfying the above equation with both u and v nonzero.

Proof. We need to show that under the assumptions given that one of the choices of u and v from part (a) have both u and v nonzero. We rename our answers to part (a) as
$$u_1 = ac-bd \text{ and } v_1 = ad+bc$$
and
$$u_2 = ac+bd \text{ and } v_2 = bc-ad$$
and we compute
$$\begin{aligned}u_1v_1 &= (ac-bd)(ad+bc) \\ &= a^2cd + abc^2 - abd^2 - b^2cd \\ &= cd(a^2-b^2) + ab(c^2-d^2)\end{aligned}$$
and
$$\begin{aligned}u_2v_2 &= (ac+bd)(bc-ad) \\ &= abc^2 + b^2cd - a^2cd - abd^2 \\ &= -cd(a^2-b^2) + ab(c^2-d^2).\end{aligned}$$
Subtracting, we have that
$$u_1v_1 - u_2v_2 = 2cd(a^2-b^2)$$
which is nonzero because $c \neq 0$, $d \neq 0$, and $a^2 \neq b^2$. But if $u_1v_1 - u_2v_2 \neq 0$ then at least one of u_1v_1 and u_2v_2 is nonzero, which in turn means that either u_1 and v_1 are both nonzero or u_2 and v_2 are both nonzero. □

(c) Assume that the integers a, b, c, and d are all nonzero, that $a^2 \neq b^2$ and that $c^2 \neq d^2$. Show that we can find two different sets $\{u^2, v^2\}$ and $\{s^2, t^2\}$ (with u, v, s, and t integers) so that

$$(a^2 + b^2)(c^2 + d^2) = u^2 + v^2 = s^2 + t^2$$

Proof. Labeling our answers from part (a) by

$$u = ac - bd \text{ and } v = ad + bc$$

and

$$s = ac + bd \text{ and } t = bc - ad$$

we seek to show that $\{u^2, v^2\}$ and $\{s^2, t^2\}$ are distinct sets. First we compute

$$\begin{aligned} u^2 - s^2 &= (u - s)(u + s) \\ &= (-2bd)(2ac) \\ &= -4abcd \end{aligned}$$

which is nonzero by the assumption that a, b, c, and d are all nonzero. Therefore $u^2 \neq s^2$. Meanwhile computing

$$\begin{aligned} u^2 - t^2 &= (u - t)(u + t) \\ &= (ac - bd - bc + ad)(ac - bd + bc - ad) \\ &= [(a - b)(c + d)][(a + b)(c - d)] \\ &= [(a - b)(a + b)][(c - d)(c + d)] \\ &= (a^2 - b^2)(c^2 - d^2) \end{aligned}$$

which is nonzero by the assumptions that $a^2 \neq b^2$ and $c^2 \neq d^2$. Therefore $u^2 \neq t^2$. This moreover allows us to conclude that $v^2 \neq s^2$ and $v^2 \neq t^2$ since otherwise we could use that

$$u^2 + v^2 = s^2 + t^2$$

to conclude that either $u^2 = s^2$ or $u^2 = t^2$. Therefore the sets $\{u^2, v^2\}$ and $\{s^2, t^2\}$ are different. $\square$

(d) Give a geometric interpretation and proof of the results in (b) and (c) above.

Proof. For part (b) we are assuming that a, b, c and d are nonzero and that $a^2 \neq b^2$. The first assumption that a, b, c and d are all nonzero implies that the real and imaginary parts of $z_1 = a + ib$ and $z_2 = c + id$ are nonzero, which geometrically means that z_1 and z_2 do not point along the direction of one of the axes. Meanwhile the assumption that $a^2 \neq b^2$ implies that $|a| \neq |b|$ which implies that $a \neq \pm b$. Geometrically this means that z_1 does not point in the diagonal directions given by the lines $x = y$ and $x = -y$. We can sum up these facts by saying that if we write z_1 and z_2 in polar form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

that θ_1 is not an integer multiple of $\pi/4$ and that θ_2 is not an integer multiple of $\pi/2$.

To show that u and v can be chosen both nonzero, we need to show the real and imaginary parts of either $z_1 z_2$ and $z_1\bar{z}_2$ are both nonzero, or geometrically that either $z_1 z_2$ or $z_1\bar{z}_2$ does not point along one of the axes. In polar coordinates we have

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \tag{1}$$
$$z_1\bar{z}_2 = r_1 r_2(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)). \tag{2}$$

If we were to assume that both $z_1 z_2$ and $z_1\bar{z}_2$ point along the axis, we'll have that

$$\theta_1 + \theta_2 = k\pi/2$$
$$\theta_1 - \theta_2 = l\pi/2$$

for some integers k and l. Adding these equations together, and dividing by 2 we find that

$$\theta_1 = (k + l)\pi/4$$

so that θ_1 is an integer multiple of $\pi/4$. This contradicts shows that at least one of $z_1 z_2$ or $z_1\bar{z}_2$ do not point along one of the axes, and therefore has nonzero both real and imaginary parts.

For part (c), we seek to show that $\{u^2, v^2\} \neq \{s^2, t^2\}$. This is equivalent to showing that $(|u|, |v|) \neq (|s|, |t|)$ and that $(|u|, |v|) \neq (|t|, |s|)$ We can argue geometrically (using similar triangles) that either of the above equalities holding is equivalent to the argument of $z_1 z_2$ differing from that of $z_1\bar{z}_2$ or that of $\overline{z_1\bar{z}_2} = \bar{z}_1 z_2$ by an integer multiple of $\pi/2$. Using (1)–(2) if the arguments of $z_1 z_2$ and $z_1\bar{z}_2$ differ by an integer multiple of $\pi/2$ we get that

$$2\theta_2 = [\theta_1 + \theta_2] - [\theta_1 - \theta_2] = k\pi/2$$

which implies that $\theta_2 = k\pi/4$ so θ_2 is an integer multiple of $\pi/4$. As discussed above, this would imply that either $c^2 = d^2$ or that $cd = 0$ in contradiction to our assumptions. Similarly, if the arguments of $z_1 z_2$ and $\bar{z}_1 z_2$ differ by an integer multiple of $\pi/2$, we use that

$$\bar{z}_1 z_2 = r_1 r_2(\cos(-\theta_1 + \theta_2) + i\sin(-\theta_1 + \theta_2))$$

to find that

$$2\theta_1 = [\theta_1 + \theta_2] - [-\theta_1 + \theta_2] = k\pi/2$$

so $\theta_1 = k\pi/4$ and hence θ_1 is an integer multiple of $\pi/4$ in contradiction to the assumptions that $a^2 \neq b^2$ and $ab \neq 0$. $\square$

(2) (1.10)

(a) Prove that

$$|z_1+z_2|^2+|z_1-z_2|^2 = 2\left(|z_1|^2+|z_2|^2\right)$$

for any complex numbers z_1, z_2.

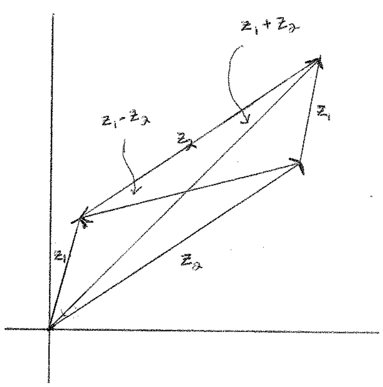
Proof. We compute:

$$\begin{aligned}
&|z_1+z_2|^2+|z_1-z_2|^2 \\
&\quad = (z_1+z_2)(\overline{z_1+z_2})+(z_1-z_2)(\overline{z_1-z_2}) && \text{definition of modulus} \\
&\quad = (z_1+z_2)(\bar{z}_1+\bar{z}_2)+(z_1-z_2)(\bar{z}_1-\bar{z}_2) && \text{prob. 7 from ch. 1} \\
&\quad = z_1\bar{z}_1+z_2\bar{z}_1+z_1\bar{z}_2+z_2\bar{z}_2+z_1\bar{z}_1-z_2\bar{z}_1-z_1\bar{z}_2+z_2\bar{z}_2 \\
&\quad = 2\left(z_1\bar{z}_1+z_2\bar{z}_2\right) \\
&\quad = 2\left(|z_1|^2+|z_2|^2\right). && \text{definition of modulus}
\end{aligned}$$

□

(b) Give a geometric interpretation of the formula in part (a) (for this part of the problem you may assume that z_1 and z_2 are both nonzero and have different arguments).

Answer. If z_1 and z_2 are both nonzero and have different arguments (i.e. point in different directions) then the parallelogram with sides z_1 and z_2 will have diagonals given by z_1+z_2 and z_1-z_2:



The identity in part (a) then corresponds to the fact from plane geometry that the sum of the squares of lengths of the diagonals of a parallelogram are equal to the sum of squares of the four side lengths of the parallelogram. □

(3) (1.16) In each part, identify the set of points which satisfy the given equation.

(a) $|z| = \text{Re}(z) + 1$

Answer. Writing in terms of x and y with $z = x + iy$, the equation becomes

$$\sqrt{x^2 + y^2} = x + 1. \tag{3}$$

Squaring both sides yields

$$x^2 + y^2 = x^2 + 2x + 1$$

which simplifies down to

$$y^2 = 2x + 1 \tag{4}$$

This is an equation of a parabola which open rightward.

Note that we need to check that every solution to (4) is also a solution to (3) since squaring both sides of an equation may introduce new solutions. Note that if a point (x, y) satifies (4) we must have $x \geq -1/2$ since $y^2 \geq 0$ always. In particular we will always have $x > -1$ for such a value of x. Substituting $y^2 = 2x + 1$ into the left-hand side of (3) we find that

$$\begin{aligned}\sqrt{x^2 + y^2} &= \sqrt{x^2 + 2x + 1} \\ &= \sqrt{(x + 1)^2} \\ &= |x + 1| \\ &= x + 1\end{aligned}$$

where in the last line we've used that $x > -1$. Therefore every solution to (4) is a solution to (3). $\square$

(b) $|z - 1| + |z + 1| = 4$

Answer. Geometrically, this equation says that the sum of the distance from z to 1 and the distance from z to -1 equals 4. The set of such z form an ellipse with foci $(-1, 0)$ and $(1, 0)$, and semi-major axis 2. The equation of such an ellipse is given by

$$\frac{x^2}{4} + \frac{y^2}{3} = 1.$$

Analytically we can prove this by substituting $z = x + iy$ and using the definition of modulus to rewrite the equation as

$$\sqrt{(x - 1)^2 + y^2} + \sqrt{(x + 1)^2 + y^2} = 4. \tag{5}$$

Since both sides are nonnegative, we can square this equation without introducing new solutions to find

$$(x - 1)^2 + y^2 + 2\sqrt{(x - 1)^2 + y^2}\sqrt{(x + 1)^2 + y^2} + (x + 1)^2 + y^2 = 16$$

which (after some tedious but straightforward computations) we can simplify to

$$\sqrt{x^4 + y^4 + 2x^2y^2 + 2y^2 - 2x^2 + 1} = 7 - x^2 - y^2.$$

Squaring both sides again leads to

$$\begin{aligned}x^4 + y^4 + 2x^2y^2 + 2y^2 - 2x^2 + 1 &= (7 - x^2 - y^2)^2 \\ &= 49 - 14x^2 - 14y^2 + 2x^2y^2 + x^4 + y^4\end{aligned}$$

which we can simplfy to

$$12x^2 + 16y^2 = 48$$

or equivalently

$$\frac{x^2}{4} + \frac{y^2}{3} = 1. \tag{6}$$

As in part (a) we need to check that every solution to (6) is also a solution to (5) since we might have introduced new solutions the second time we squared the equation. We first note that if (x, y) solves (6) we must have that $x \in (-2, 2)$ since we have that

$$\frac{x^2}{4} - 1 = -\frac{y^2}{3} \leq 0 \quad \Rightarrow \quad x^2 \leq 4 \quad \Rightarrow \quad -2 \leq x \leq 2.$$

Substituting $y^2 = 3(1 - x^2/4)$ into the left-hand-side of (5) then leads to

$$\begin{aligned}
\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} &= \sqrt{(x-1)^2 + 3(1 - x^2/4)} + \sqrt{(x+1)^2 + 3(1 - x^2/4)} \\
&= \sqrt{\frac{x^2}{4} - 2x + 4} + \sqrt{\frac{x^2}{4} + 2x + 4} \\
&= \sqrt{\left(\frac{x}{2} - 2\right)^2} + \sqrt{\left(\frac{x}{2} + 2\right)^2} \\
&= \left|\frac{x}{2} - 2\right| + \left|\frac{x}{2} + 2\right|
\end{aligned}$$

and using that $x \in (-2, 2)$, we conclude that $\frac{x}{2} - 2 < 0$ and that $\frac{x}{2} + 2 > 0$ so we can continue

$$\begin{aligned}
&= -\left(\frac{x}{2} - 2\right) + \left(\frac{x}{2} + 2\right) \\
&= 4.
\end{aligned}$$

Thus every solution to (6) is also a solution to (5). □

(c) $z^{n-1} = \bar{z}$ (where n is an integer)

Answer. We consider the equation in the following cases seperately: $n = 0$, $n = 1$, $n = 2$, $n \geq 3$, and $n \leq -1$.

We first consider the case that $n = 2$. In this case the equation reduces to

$$z = \bar{z} \quad \Longleftrightarrow \quad z - \bar{z} = 0.$$

Writing $z = x + iy$, this becomes

$$(x + iy) - (x - iy) = 0 \quad \Longleftrightarrow \quad 2iy = 0 \quad \Longleftrightarrow \quad y = 0.$$

Thus in the case that $n = 2$ the set of solutions to the equation is the real axis (this should be geometrically clear from the equation $z = \bar{z}$).

We next consider the case $n = 0$. Here the equation becomes $1 = \bar{z}$ which is true precisely when $z = 1$.

We next consider $n = 0$, i.e. $z^{-1} = \bar{z}$. Here we must have $z \neq 0$ or else the left-hand side is not defined. Multiplying both sides by z, turns the equation into $1 = z\bar{z} = |z|^2$, so the solution set is precisely the set of complex numbers with modulus 1. Therefore in the case $n = 0$, we find that the solution set the equation is the unit circle.

We next consider the case $n \geq 3$. In this case $z = 0$ is solution. For $z \neq 0$ we can take the modulus of each side of the equation to find that

$$z^{n-1} = \bar{z} \quad \Rightarrow \quad \left|z^{n-1}\right| = |\bar{z}| \quad \Rightarrow \quad |z|^{n-1} = |z| \quad \Rightarrow \quad |z|^n = 1 \quad \Rightarrow \quad |z| = 1$$

so a nonzero solution must have modulus 1. Multiplying both sides of the original equation $z^{n-1} = \bar{z}$ by $z \neq 0$, we find that $z \neq 0$ is a solution precisely when

$$z^n = z\bar{z} = |z|^2 = 1,$$

i.e. precisely when z is an n-root of unity. Therefore the solutions in this case are $z = 0$ and $z = \cos(2\pi k/n) + i\sin(2\pi k/n)$ for $k \in \{0, 1, \ldots, n-1\}$.

We finally consider the case $n \leq -1$. In this case that left hand side of the equation is not defined so we are therefore free to assume that $z \neq 0$. Now, arguing as in the previous paragraph, we have that

$$z^{n-1} = \bar{z} \quad \Rightarrow \quad \left|z^{n-1}\right| = |\bar{z}| \quad \Rightarrow \quad |z|^{n-1} = |z| \quad \Rightarrow \quad |z|^n = 1 \quad \Rightarrow \quad |z| = 1$$

so a solution must have modulus 1. Again arguing as in the previous paragraph, we multiply both sides by z to find that z solves $z^{n-1} = \bar{z}$ precisely when it solves

$$z^n = 1$$

(with z now negative). The solutions to this equation are precisely the $|n|$-th roots of unity, i.e. $z = \cos(2\pi k/|n|) + i\sin(2\pi k/|n|)$ for $k \in \{0, 1, \ldots, |n|-1\}$. $\square$

(4) (2.2) Let f be a complex-valued function.

(a) Assume for all purely real z that $f(z)$ is purely real and differentiable. Show that $f'(z)$ is also purely real for all purely real z.

Proof. Let $z = x$ with x real. By assumption, f is differentiable at x, so the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. Using the sequential characterization of limits, this implies that

$$f'(x) = \lim_{n \to \infty} \frac{f(x+h_n) - f(x)}{h_n}$$

for any sequence h_n converging to zero. Letting $h_n = 1/n$, then $f(x + 1/n)$ is purely real for all n since $x + 1/n$ is real. Therefore

$$\frac{f(x+1/n) - f(x)}{1/n}$$

is purely real for all $n \in \mathbb{N}$ (since differences and quotients of purely real quantities are purely real) and hence

$$f'(x) = \lim_{n \to \infty} \frac{f(x+1/n) - f(x)}{1/n}$$

is purely real since

$$\operatorname{Im}(f'(x)) = \lim_{n \to \infty} \operatorname{Im}\left(\frac{f(x+1/n) - f(x)}{1/n}\right) = \lim_{n \to \infty} 0 = 0.$$

$\square$

(b) Assume for all purely imaginary z that $f(z)$ is purely real and differentiable. Show that $f'(z)$ is purely imaginary for all purely imaginary z.

Proof. Let $z = iy$ with y real. By assumption, f is differentiable at iy, so the limit

$$f'(iy) = \lim_{h \to 0} \frac{f(iy+h) - f(iy)}{h}$$

exists. Using the sequential characterization of limits, this implies that

$$f'(iy) = \lim_{n \to \infty} \frac{f(iy+h_n) - f(iy)}{h_n}$$

for any sequence h_n converging to zero. Letting $h_n = i/n$, then $f(iy + i/n)$ is purely real for all n since $i(y + 1/n)$ is purely imaginary. Therefore, using that $1/i = -i$, we can see that

$$\frac{f(iy+i/n) - f(iy)}{i/n} = -i\frac{f(iy+i/n) - f(iy)}{1/n}$$

is purely imaginary (since it is of the from $i \cdot$ (purely real)) and hence

$$f'(iy) = \lim_{n \to \infty} \frac{f(iy+i/n) - f(iy)}{i/n}$$

is purely imaginary since

$$\operatorname{Re}(f'(iy)) = \lim_{n \to \infty} \operatorname{Re}\left(\frac{f(iy+i/n) - f(iy)}{i/n}\right) = \lim_{n \to \infty} 0 = 0.$$

$\square$

(5) (a) Assume that $f : \mathbb{C} \to \mathbb{C}$ is differentiable everywhere and that $f(z)(= f(x + iy))$ is purely real along both of the lines $x = a$ and $y = b$ for some real constants a and b. Prove that $f'(a+ib) = 0$.

Proof. First assume that $a = b = 0$, i.e. that f is purely real along both the real and imaginary axes. Then problem 4(a) lets us conclude that $f'(0)$ is purely real, while problem 4(b) lets us conclude that $f'(0)$ is purely imaginary. Since 0 is the only number that is both purely real and purely imaginary, it must be the case that $f'(0) = 0$.

Now, to address the general case, we define $g(z) = f(z + a + ib)$. Then g is differentiable everywhere since

$$\lim_{h\to 0} \frac{g(z+h) - g(z)}{h} = \lim_{h\to 0} \frac{f(z + a + ib + h) - f(z + a + ib)}{h} \tag{7}$$

and the limit on the right exists by the assumption that f is differentiable everywhere. Moreover, g is real along the real axis, since for x real we have $g(x) = f(x+a+ib)$ and f is assumed to be real along the line $y = b$. Similarly g is real along the imaginary axis, since for y real we have that $g(iy) = f(a + i(b + y))$ and f is assumed to real along the line $x = a$.

We can now apply the argument of the first paragraph to g and conclude that $g'(0) = 0$, using (7), we can conclude that $f'(a + ib) = g'(0) = 0$. □

(b) Assume that $f : \mathbb{C} \to \mathbb{C}$ is differentiable everywhere and that $f(z)$ is purely real for all $z \in \mathbb{C}$. Show that $f'(z) = 0$ for all $z \in \mathbb{C}$.

Proof. Since $f(z)$ is real everywhere, it is real along the lines $x = a$ and $y = b$ for any real numbers a and b. Therefore, applying part (a) to f, we can conclude that $f(a + ib) = 0$ for all a and b. □

(c) Assume that $f : \mathbb{C} \to \mathbb{C}$ is differentiable everywhere and that $f(z)$ is purely imaginary for all $z \in \mathbb{C}$. Show that $f'(z) = 0$ for all $z \in \mathbb{C}$.

Proof. Define $g(z) = if(z)$. Then $g(z)$ is purely for all $z \in \mathbb{C}$ since $f(z)$ is purely imaginary (f must be of the form $f(z) = iv(z)$ with v real, so $g(z) = if(z) = i^2v(z) = -v(z)$). Moreover, since f is differentiable everywhere and

$$\lim_{h\to 0} \frac{g(z+h) - g(z)}{h} = \lim_{h\to 0} \frac{if(z+h) - if(z)}{h} = i \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

we can conclude that $g(z)$ is differentiable everywhere and $g'(z) = if'(z)$. Applying part (b) to $g(z)$, we can conclude that $g'(z) = 0$ for all $z \in \mathbb{C}$, and therefore that $f'(z) = g'(z)/i = 0/i = 0$ for all $z \in \mathbb{C}$. □

(6) (2.15) Find the radius of convergence of each of the following series.

(a) $\sum_{n=0}^{\infty} \sin(n) z^n$

Proof. We will argue that

$$\limsup |\sin(n)|^{1/n} = 1$$

in a series of claims. It will then follow from Theorem 2.8 that the radius of convergence of $\sum_{n=0}^{\infty} \sin(n) z^n$ is $1/1 = 1$.

Claim 1. For any $x \in \mathbb{R}$, if $|\sin x| \leq 1/2$, then $|\sin(x+2)| > 1/2$.

Proof of Claim 1. If $|\sin x| \leq 1/2$ then there exists an integer k so that

$$-\pi/6 \leq x - k\pi \leq \pi/6.$$

Using that $3 < \pi$, we have that $2 < \frac{2\pi}{3}$, and using that $\pi < 4$ we have that $\frac{\pi}{2} < 2$, so

$$\frac{\pi}{2} < 2 < \frac{2\pi}{3}.$$

Adding this inequality to the previous one, we can conclude that

$$-\pi/6 + \pi/2 < x + 2 - k\pi < \pi/6 + 2\pi/3$$

or

$$\pi/3 < x + 2 - k\pi < 5\pi/6$$

which implies that $|\sin(x+2)| > \frac{1}{2}$. $\square$

Claim 2. For any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ so that for $n \geq N$, $|\sin(n)|^{1/n} < 1 + \varepsilon$.

Proof of Claim 2. We have for any $n \in \mathbb{N}$ that $0 \leq |\sin n| \leq 1$, from which it follows that $|\sin n|^{1/n} \leq 1^{1/n} = 1$ for all $n \in \mathbb{N}$, and therefore $|\sin n|^{1/n} < 1 + \varepsilon$ for $n \in \mathbb{N}$ and $\varepsilon > 0$. $\square$

Claim 3. For any $\varepsilon > 0$ and any $N \in \mathbb{N}$ there is an $n \geq N$ so that $|\sin n|^{1/n} > 1 - \varepsilon$.

Proof of Claim 3. Let $\varepsilon > 0$ and let $N \in \mathbb{N}$. Since $\lim_{n\to\infty} (1/2)^{1/n} = 1$, we can find an $M \in \mathbb{N}$ with $M \geq N$ so that for all $n \geq M$,

$$(1/2)^{1/n} > 1 - \varepsilon.$$

But then it follows from Claim 1 that either $|\sin(M)| > 1/2$ or $|\sin(M+2)| > 1/2$, which in turn implies that either

$$|\sin(M)|^{1/M} > (1/2)^{1/M} > 1 - \varepsilon$$

or

$$|\sin(M+2)|^{1/(M+2)} > (1/2)^{1/(M+2)} > 1 - \varepsilon.$$

In either case, we have found an $n \geq N$ so that $|\sin(n)|^{1/n} > 1 - \varepsilon$. $\square$

To finish the proof, we merely observe that according to a theorem stated in lecture, the statement $\limsup |\sin(n)|^{1/n} = 1$ is equivalent to both Claims 2 and 3 being true. Therefore $\limsup |\sin(n)|^{1/n} = 1$ and the radius of convergence of the power series $\sum_{n=0}^{\infty} \sin(n) z^n$ is 1. $\square$

(b) $\sum_{n=0}^{\infty} e^{-n^2} z^n$

Proof. We compute

$$\lim_{n\to\infty} \left| e^{-n^2} \right|^{\frac{1}{n}} = \lim_{n\to\infty} e^{-n} = 0.$$

Therefore according to Theorem 2.8, the radius of convergence of the power series is $+\infty$. $\square$