**Math 425, Homework #1**
**Due: February 7, 2011**

Instructions:

- Write in compete sentences, organized into paragraphs.
- Leave plenty of room in between problems.
- Write only on the front side of each sheet of paper.
- Staple!
- Write on your assignment the names of any persons or sources consulted during its completion (other than the course text or instructor).

(1) (1.9)

(a) Use complex algebra to show that for any four integers $a$, $b$, $c$, and $d$ there are integers $u$ and $v$ so that

$$(a^2 + b^2)(c^2 + d^2) = u^2 + v^2$$

(b) Assume that the integers $a$, $b$, $c$, and $d$ are all nonzero and that $a^2 \neq b^2$. Show that we can find integers $u$ and $v$ satisfying the above equation with both $u$ and $v$ nonzero.

(c) Assume that the integers $a$, $b$, $c$, and $d$ are all nonzero, that $a^2 \neq b^2$ and that $c^2 \neq d^2$. Show that we can find two different sets $\{u^2, v^2\}$ and $\{s^2, t^2\}$ (with $u$, $v$, $s$, and $t$ integers) so that

$$(a^2 + b^2)(c^2 + d^2) = u^2 + v^2 = s^2 + t^2$$

(d) Give a geometric interpretation and proof of the results in (b) and (c) above.

(2) (1.10)

(a) Prove that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$$

for any complex numbers $z_1$, $z_2$.

(b) Give a geometric interpretation of the formula in part (a) (for this part of the problem you may assume that $z_1$ and $z_2$ are both nonzero and have different arguments).

(3) (1.16) In each part, identify the set of points which satisfy the given equation.

(a) $|z| = \mathrm{Re}(z) + 1$

(b) $|z - 1| + |z + 1| = 4$

(c) $z^{n-1} = \bar{z}$ (where $n$ is an integer)

(4) (2.2) Let $f$ be a complex-valued function.

(a) Assume for all purely real $z$ that $f(z)$ is purely real and differentiable. Show that $f'(z)$ is also purely real for all purely real $z$.

(b) Assume for all purely imaginary $z$ that $f(z)$ is purely real and differentiable. Show that $f'(z)$ is purely imaginary for all purely imaginary $z$.

(5) (a) Assume that $f : \mathbb{C} \to \mathbb{C}$ is differentiable everywhere and that $f(z)(= f(x + iy))$ is purely real along both of the lines $x = a$ and $y = b$ for some real constants $a$ and $b$. Prove that $f'(a+ib) = 0$.

(b) Assume that $f : \mathbb{C} \to \mathbb{C}$ is differentiable everywhere and that $f(z)$ is purely real for all $z \in \mathbb{C}$. Show that $f'(z) = 0$ for all $z \in \mathbb{C}$.

(c) Assume that $f : \mathbb{C} \to \mathbb{C}$ is differentiable everywhere and that $f(z)$ is purely imaginary for all $z \in \mathbb{C}$. Show that $f'(z) = 0$ for all $z \in \mathbb{C}$.

(6) (2.15) Find the radius of convergence of each of the following series.

(a) $\sum_{n=0}^{\infty} \sin(n) z^n$

(b) $\sum_{n=0}^{\infty} e^{-n^2} z^n$