1. Evaluate the following line integrals.

   (a) $\int_C \bar{z}\,dz$, where C is the straight-line segment connecting 0 to $2+2i$.

   Answer. The curve C can be parametrized as

$$z(t) = t(2+2i) \qquad t \in [0,1].$$

   We note that $z'(t) = 2+2i$. Using the definition of line integral, we then find that

$$\begin{aligned}\int_C \bar{z}\,dz &:= \int_0^1 \bar{z}(t)z'(t)\,dt \\ &= \int_0^1 t(2-2i)(2+2i)\,dt \\ &= \int_0^1 8t\,dt \\ &= \left[4t^2\right]_{t=0}^1 = 4.\end{aligned}$$

□

   (b) $\int_C z\cos^2 z\,dz$, where C is the boundary of the triangle with vertices 0, i, and $1+i$ traversed once around in the counter-clockwise direction.

   Answer. The function $f(z) = z\cos^2 z$ is an entire function (since $\cos z$ and z are entire and products of entire functions are entire). Since the curve is closed, the closed curve theorem implies that

$$\int_C z\cos^2 z\,dz = 0.$$

□

2. (a) State the Cauchy Integral Formula. In your statement, use **complete sentences** to explain all notation and any assumptions which are necessary for the Cauchy Integral Formula to be true.

Answer. Let D be a closed disk, C the boundary circle of the disk traversed once in the counterclockwise direction, g an analytic function on the disk D, and a a point in the interior of D. Then the Cauchy Integral Formula says that

$$g(a) = \frac{1}{2\pi i} \int_C \frac{g(z)}{z-a}\, dz.$$

□

(b) Let $f(z) = \frac{z^2}{z^2+2z+2}$. Evaluate the line integral $\int_C f(z)\, dz$ where C is the smooth curve defined by:

Before answering the individual questions, we rewrite the function f by factoring the denominator. Using the quadratic formula, we find that the zeroes of $z^2 + 2z + 2$ occur at

$$z = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = -1 \pm i$$

so we find that $z^2 + 2z + 2 = (z - [-1+i])(z - [-1-i])$, and hence the function f can be rewritten $f(z) = \frac{z^2}{(z-[-1+i])(z-[-1-i])} = \frac{z^2}{(z+1-i)(z+1+i)}$.

i. $z(t) = i + 2e^{it} \qquad t \in [0, 2\pi]$

Answer. The curve in question here is a circle of radius 2 centered at i travelled once in the counterclockwise direction. Computing the distance from $-1 \pm i$ to i we see that the point $-1+i$ is in the interior of the disk enclosed by C, and the point $-1-i$ is outside the disk enclosed by C. Thus the function $g(z) = \frac{z^2}{z+1+i}$ is analytic on the closed disk enclosed by C so we can apply the Cauchy Integral Formula with $a = -1+i$ to find that

$$\begin{aligned}\int_C \frac{z^2}{z^2+2z+2}\, dz &= \int_C \frac{z^2/(z+1+i)}{z-(-1+i)}\, dz = \int_C \frac{g(z)}{z-(-1+i)}\, dz \\ &= 2\pi i g(-1+i) = 2\pi i \frac{(-1+i)^2}{-1+i+1+i} = \pi(-1+i)^2 \\ &= -2\pi i.\end{aligned}$$

□

ii. $z(t) = -1 - i + e^{-i2t} \quad t \in [0, 2\pi]$

Answer. The curve is the boundary of a disk or radius 1 centered at $-1-i$ traversed twice in the clockwise direction. The point $-1-i$ is clearly in the interior of the disk (since it's the center) while the point $-1+i$ is distance two to the center so it is outside the disk. Thus the function $g(z) = \frac{z^2}{z+1-i}$ is analytic on the closed disk enclosed by C. We can therefore apply the Cauchy Integral Formula with $a = -1-i$ and with C' the boundary of this disk traversed once in the counterclockwise direction to find:

$$\begin{aligned}\int_{C'} \frac{z^2}{z^2+2z+2}\, dz = \int_{C'} \frac{z^2/(z+1-i)}{z-(-1-i)} &= \int_{C'} \frac{g(z)}{z-(-1-i)} \\ &= 2\pi i g(-1-i) = 2\pi i \frac{(-1-i)^2}{-1-i+1-i} \\ &= -\pi(-1-i)^2 = -2\pi i.\end{aligned}$$

Thus for the curve C (which traces C' twice in the opposite direction) we need to multiply the above by -2 to get

$$\int_C \frac{z^2}{z^2+2z+2}\, dz = 4\pi i.$$

□

3. Suppose that the power series $\sum_{k=0}^{\infty} a_k(z-4)^k$ satisfies

$$\sum_{k=0}^{\infty} a_k(z-4)^k = \frac{\cos z}{z^2+9}$$

for all z in some open set containing $z = 4$. Find the radius of convergence of this power series, and explain why you know your answer is correct.

Proof. We saw in class that a function can be written as a convergent power series centered at $z = 4$ on the largest disk centered at $z = 4$ on which the function is analytic. Since $\cos z$ and $z^2 + 9$ are entire functions, $\frac{\cos z}{z^2+9}$ will be analytic wherever the denominator is nonzero, i.e. for all $z \neq \pm 3i$. Computing the distance from 4 to $\pm 3i$, we find $|4 - \pm 3i| = \sqrt{4^2 + 3^2} = 5$ so $\frac{\cos z}{z^2+9}$ is analytic on an open disk of radius 5 centered at $z = 4$. Therefore, there exists a power series $\sum_{k=0}^{\infty} b_k(z-4)^k$ with radius of convergence equal to 5 and with

$$\frac{\cos z}{z^2+9} = \sum_{k=0}^{\infty} b_k(z-4)^k$$

for all z with $|z - 4| < 5$. By the assumption that

$$\sum_{k=0}^{\infty} a_k(z-4)^k = \frac{\cos z}{z^2+9}$$

on some open set containing $z = 4$, the uniqueness theorem for power series tells us that $a_k = b_k$ for all k. Therefore the radius of convergence of $\sum_{k=0}^{\infty} a_k(z-4)^k$ is 5. □

4. Suppose that f is an entire function satisfying

$$|f(z)| \leq |z|^5$$

for all $z \in \mathbb{C}$. Show that the k-th derivative $f^{(k)}(z)$ satisfies $f^{(k)}(0) = 0$ for all $k \geq 6$. (Recall that since f is entire

$$f^{(k)}(0) = \frac{k!}{2\pi i} \int_C \frac{f(z)}{z^{k+1}}\, dz$$

where C is a circle centered at 0 traversed once in the counter-clockwise direction. What happens as the radius of C gets large?)

Proof. Let C_R denote the circle of radius R centered at 0. The assumption that $|f(z)| \leq |z|^5$ for all z implies that $|f(z)| \leq R^5$ on C_R, and hence

$$\left| \frac{f(z)}{z^k} \right| \leq \frac{R^5}{R^{k+1}} = R^{4-k}$$

for any z on C_R. Using the ML-formula with the fact that the arclength of C_R is $2\pi R$, we then get that

$$\left| f^{(k)}(0) \right| = \left| \frac{k!}{2\pi i} \int_{C_R} \frac{f(z)}{z^{k+1}}\, dz \right| = \frac{k!}{2\pi} \left| \int_{C_R} \frac{f(z)}{z^{k+1}}\, dz \right| \leq \frac{k!}{2\pi} R^{4-k}(2\pi R) = k! R^{5-k}$$

If $k \geq 6$, then $5 - k \leq -1 < 0$ so we find that

$$0 \leq \left| f^{(k)}(0) \right| = \lim_{R\to\infty} \left| f^{(k)}(0) \right| \leq \lim_{R\to\infty} k! R^{5-k} = 0$$

We conclude that $\left| f^{(k)}(0) \right| = 0$ and hence $f^{(k)}(0) = 0$ for all $k \geq 6$. $\square$

5. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous (but not necessarily analytic) function, and for $z \in \mathbb{C}$ let C_z be the smooth curve defined by

$$z(t) = tz \quad t \in [0,1],$$

so that C_z is the straight line segment connecting 0 and z. Define a function $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \int_{C_z} f(w)\, dw.$$

Prove that $\lim_{z\to 0} \frac{F(z)}{z} = f(0)$.

Proof. Let $\varepsilon > 0$. Since f is continuous, there is a $\delta > 0$ so that for $0 < |z| < \delta$, $|f(z) - f(0)| < \varepsilon$. Then for $|z| < \delta$, we have that

$$\begin{aligned}
\left|\frac{F(z)}{z} - f(0)\right| &= \left|\frac{1}{z}\int_{C_z} f(w)\, dw - f(0)\right| \\
&= \left|\frac{1}{z}\int_{C_z} f(w)\, dw - f(0)\frac{1}{z}\int_{C_z} 1\, dw\right| && \text{since } \int_{C_z} 1\, dw = [w]_{w=0}^{z} = z \\
&= \left|\frac{1}{z}\int_{C_z} f(w) - f(0)\, dw\right| \\
&= \frac{1}{|z|}\left|\int_{C_z} f(w) - f(0)\, dw\right| \\
&\le \frac{1}{|z|}\varepsilon\, \text{arclength}(C_z) && ML\text{-formula with } |f(w) - f(0)| < \varepsilon \text{ on } C_z \\
&= \frac{1}{|z|}\varepsilon\, |z| = \varepsilon.
\end{aligned}$$

We conclude that $\lim_{z\to 0} \frac{F(z)}{z} = f(0)$. □

Alternate proof. Write $f(z) = u(z) + iv(z)$ with $u(z) = \text{Re}(f(z))$ and $v(z) = \text{Im}(f(z))$. Using the definition of line integral we have that

$$\begin{aligned}
\frac{F(z)}{z} &= \frac{1}{z}\int_{C_z} f(w)\, dw \\
&= \frac{1}{z}\int_0^1 f(tz)z\, dt \\
&= \int_0^1 f(tz)\, dt \\
&= \int_0^1 u(tz)\, dt + i\int_0^1 v(tz)\, dt.
\end{aligned}$$

Since u and v are continuous, the mean value theorem for integrals from calculus tells us there are point s_z, $t_z \in [0, 1]$ so that

$$\int_0^1 u(tz)\,dt = (1-0)u(s_z z) = u(s_z z) \qquad \text{and} \qquad \int_0^1 v(tz)\,dt = v(t_z z),$$

so we can write

$$\frac{F(z)}{z} = u(s_z z) + iv(t_z z).$$

Letting $z \to 0$, we have that $s_z z \to 0$ and $t_z z \to 0$ since s_z, $t_z \in [0, 1]$ so, by continuity of u and v, we conclude that

$$\lim_{z \to 0} \frac{F(z)}{z} = u(0) + iv(0) = f(0).$$

□